# From Topological Defects to Quantum Entanglement

*A Unified Perspective via Symbols, Qubits, and Geometric Phases*

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SPICE Workshop 05/18/26

# Talk Roadmap

**Setting the Stage** The central question: are topology and entanglement fundamentally linked?

**Symbol & Topological Defects in Graphene** The Weyl symbol $H(k,r)$, defect dimension $D$, graphene vacancies, index theorem

**Two Qubits Go Topological** Cartan decomposition, $Z_5$ phases, Bell-state zero modes

**Topological Sum Rule** Geometric phases, SWAP vs 3-CNOT, noise measurement

**The Unified Picture** One framework connecting defects, qubits, and quantum gates

# Two Apparently Unrelated Questions

?

**Can a crystal defect be topological?**

*ACT I → Graphene vacancies and the Weyl symbol operator*

?

**Is quantum entanglement a topological property?**

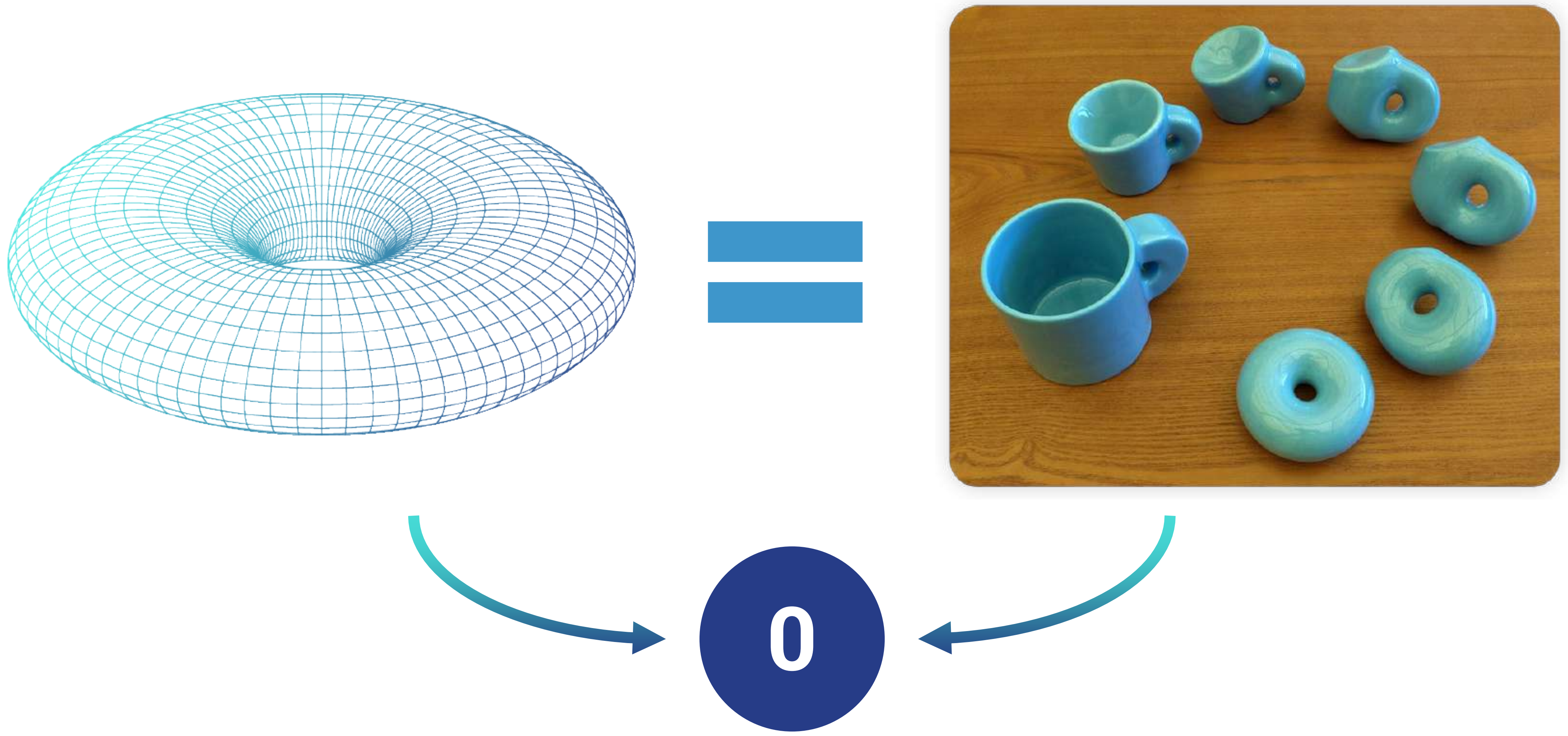
*ACT II → Two-qubit Hilbert space and Cartan decomposition*

**Answer: YES to both — and they are connected by the same mathematics.**

# Topological features :
# a general perspective

Speaking of topology requires a clear framework

# Topology in a nutshell



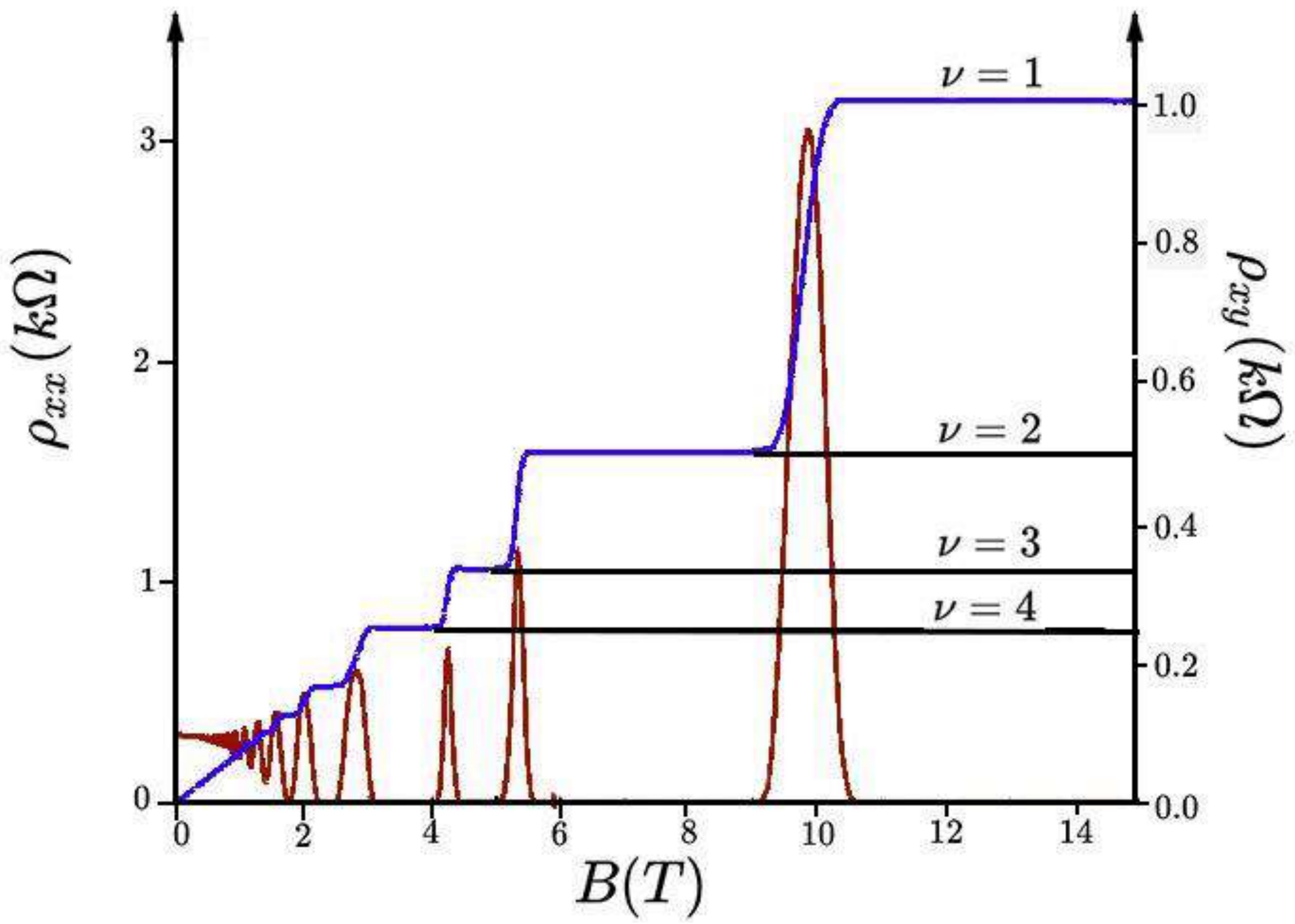

Classifies “objects” by assigning a single integer to each topological family

$2(1-\#\text{holes})=\mathbb{Z}$ = integer numbers

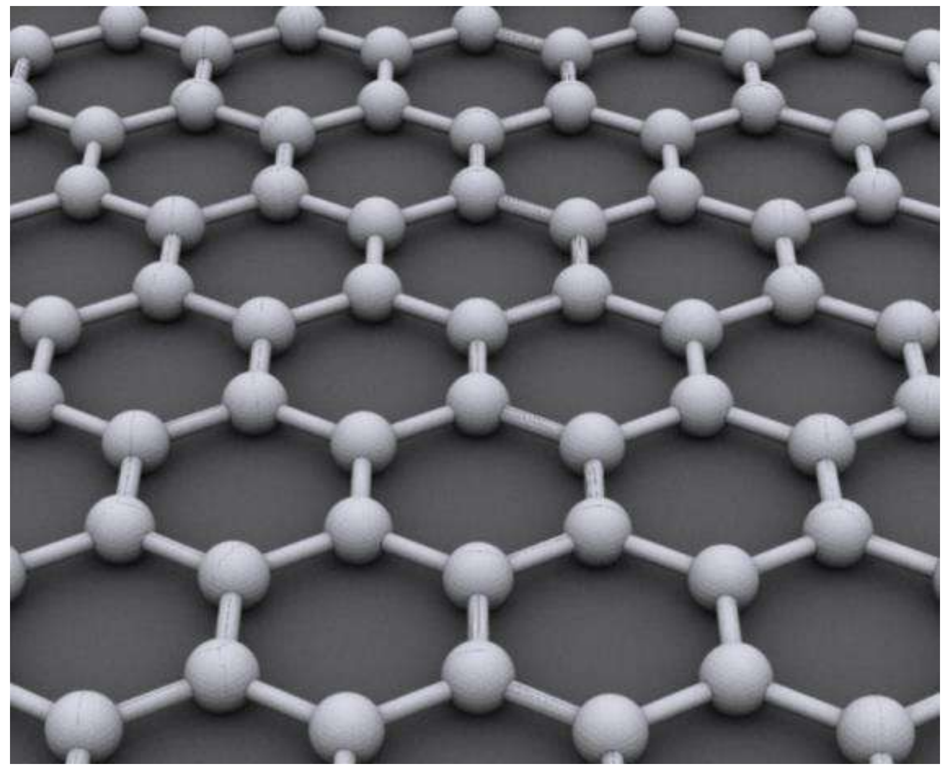
# Topology in Crystals
(2008)

## Iconic examples

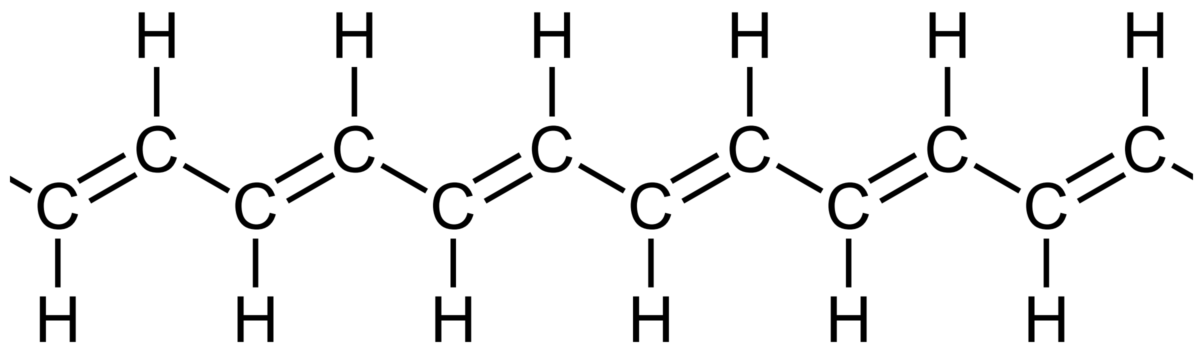
**Quantum Hall effect**

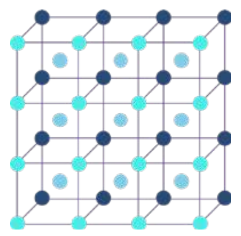



**Graphene**



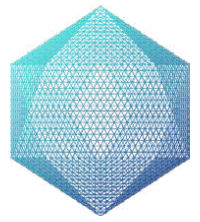
**Polyacetylene (SSH model)**



# Topology in Crystals 

(2008)

**For translation invariance/crystals** →

Classified by Brillouin zones 

Assigning integers to **topological Bloch Hamiltonian**

**Quantum transport**
IQHE
Berry curvature / phase

**Protected edge states**
Majorana modes

**Quantum computing**

# Tenfold classification

**Kitaev** (2008) **Altland** and **Zirnbauer** (1997) **Schnyder** et al (2008)

# Wigner-Dyson classification (d=0).

Remaining symmetries of a random Hamiltonian once all **unitary** symmetries have been exhausted (Anderson localisation).

Time reversal symmetry T

| Name (Cartan) | T |
|---|---|
| A (unitary) | 0 |
| AI (orthogonal) | +1 |
| AII (symplectic) | -1 |

A (0) ← Magnetic Field/Aharonov-Bohm magnetic flux

AII (-1) ← Time reversal in the presence of spin-orbit

# Altland & Zirnbauer (1997) : Extend the Wigner-Dyson classification (d=0).

| Name (Cartan) | T | C | S= T C |
|---|---|---|---|
| A (unitary) | 0 | 0 | 0 |
| AI (orthogonal) | +1 | 0 | 0 |
| AII (symplectic) | -1 | 0 | 0 |
| AIII (chiral unitary) | 0 | 0 | 1 |
| BDI (chiral orth.) | +1 | +1 | 1 |
| CII (chiral sympl.) | -1 | -1 | 1 |
| D | 0 | +1 | 0 |
| C | 0 | -1 | 0 |
| DIII | -1 | +1 | 1 |
| CI | +1 | -1 | 1 |

Time reversal symmetry (T)
Particle-Hole symmetry (C)
Their product S =T C (chiral symmetry)

Anti-unitary symmetries !

Elegant mathematical structure inherited from Cartan classification.

# The Tenfold classification in Crystals (2008)

**Two anti-unitary symmetries:**

"Time reversal" & "Particle-Hole"

d: space dimension

Tenfold: 10 classes

| Class | s | $T$ | $P$ | $C$ | $d=0$ | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| AIII | 1 | 0 | 0 | 1 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AI | 0 | + | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| BDI | 1 | + | + | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| D | 2 | 0 | + | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| DIII | 3 | - | + | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| AII | 4 | - | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| CII | 5 | - | - | 1 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| C | 6 | 0 | - | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 |
| CI | 7 | + | - | 1 | 0 | 0 | 0 | $2\mathbb{Z}$ |

Cartan Classification of Lie groups

**Kitaev** (2008) **Altland** and **Zirnbauer** (1997) **Schnyder** et al (2008)

# The Tenfold classification in Crystals (2008)

## Reading the tenfold table

A crystal having both "**Time reversal**" and "**Particle-Hole**" in **d=1** is **topological**

| Class | s | $T$ | $P$ | $C$ | $d$=0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| AIII | 1 | 0 | 0 | 1 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AI | 0 | + | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| BDI | 1 | + | + | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| D | 2 | 0 | + | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| DIII | 3 | - | + | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| AII | 4 | - | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
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Kitaev (2008) Altland and Zirnbauer (1997) Schnyder et al (2008)

# The Tenfold classification in Crystals

Relates theory to real materials

Synthesis and hybridisation of new materials

| Class | $s$ | $T$ | $P$ | $C$ | $d=0$ | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| AIII | 1 | 0 | 0 | 1 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AI | 0 | + | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| BDI | 1 | + | + | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| D | 2 | 0 | + | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| DIII | 3 | - | + | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| AII | 4 | - | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
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| C | 6 | 0 | - | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 |
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**SSH**
(conducting polymers)

**Hybrid nanowires**
(InSb)
Kitaev chain

**IQHE**
(2d semiconductors)

**Graphene**
(not topological)

**p-wave superconductors**
(metallic oxide $Sr_2RuO_4$)

**TMDC $WTe_2$**
Kane Mele

Empty !?

A haphazardly search for topological materials

# Topology lives in the symbol, not the Hamiltonian.

Defect dimension D navigates the tenfold table · Index theorem: ν zero modes protected by topology · Observable by STM

# The Problem with Bloch Hamiltonians

**Bloch's theorem works beautifully when translation symmetry holds:**
H(k) encodes band topology in the Brillouin zone. But...

**H(k) → band topology in the Brillouin zone**

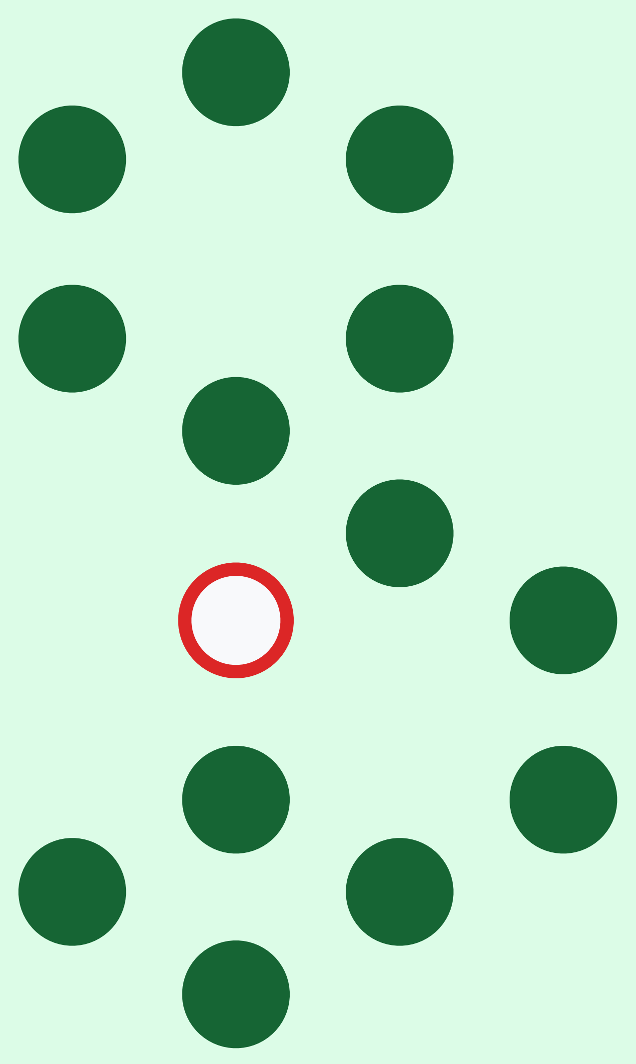
A defect — vacancy, adatom, Kekule distortion — breaks translation symmetry.

**Naive classification:** graphene ∈ BDI class, d = 2 → no Z-topology.

**This conclusion is wrong.** Understanding why is the entry point.

## Defects ?

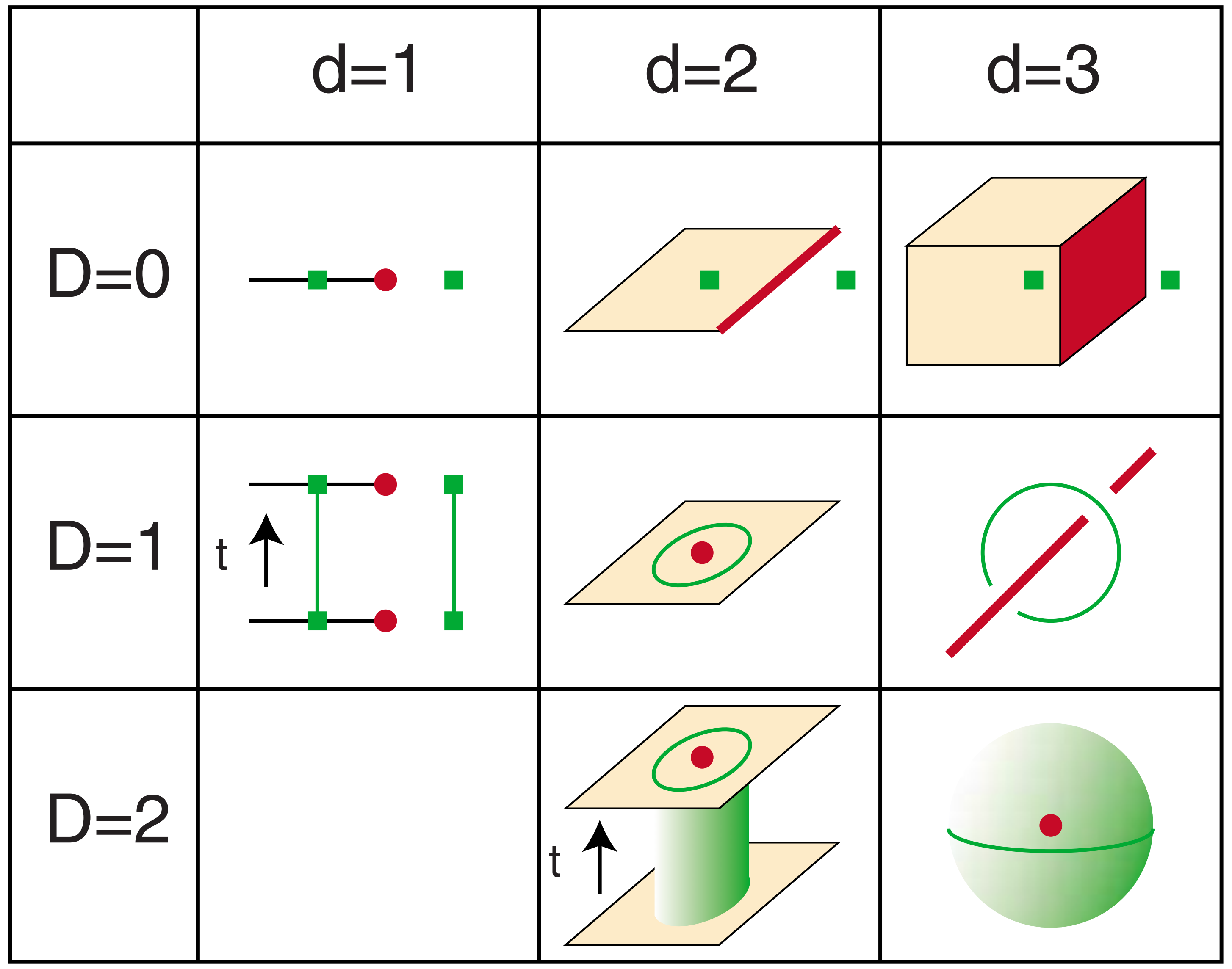
**Graphene honeycomb**

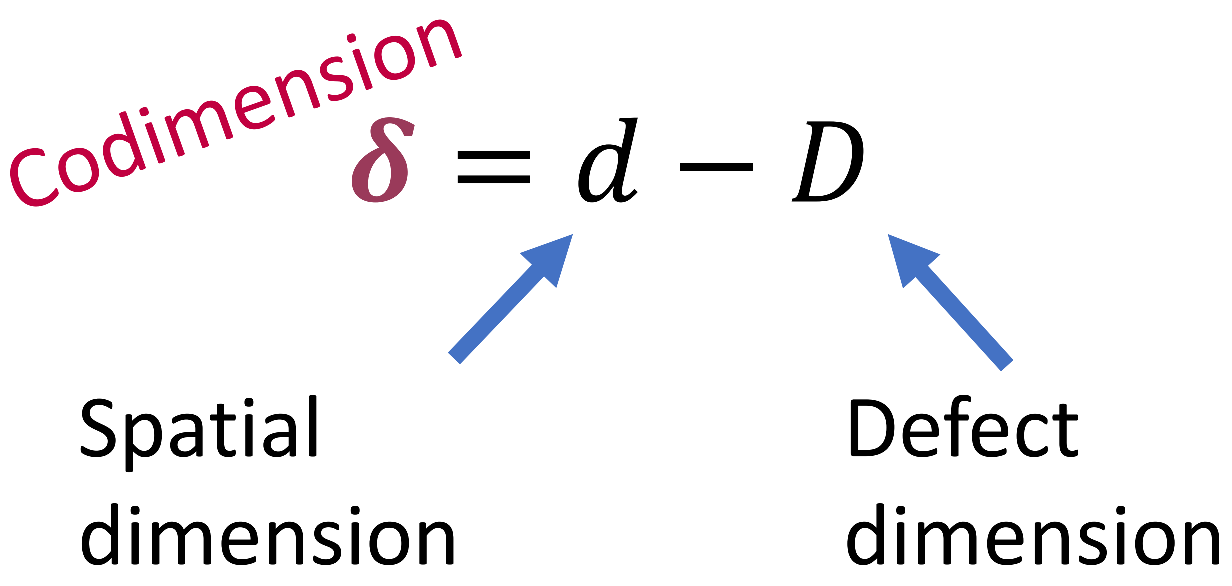


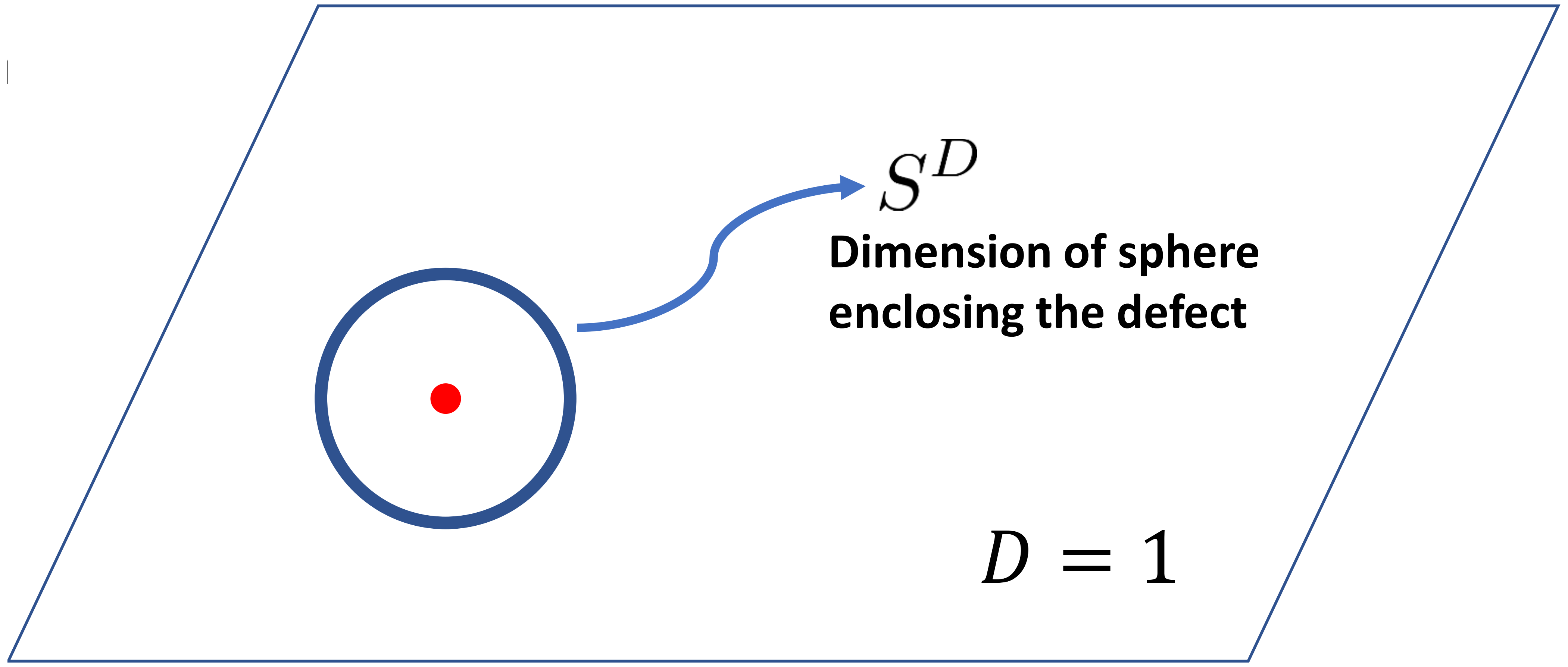
*vacancy (k not good q. n.)*

**Naive tenfold:**
**BDI, d=2 → No topology??**

# Defects in topological crystals




Teo and Kane (2010)

# Defects in topological crystals

**Two anti-unitary symmetries:**
"Time reversal" & "Particle-Hole"

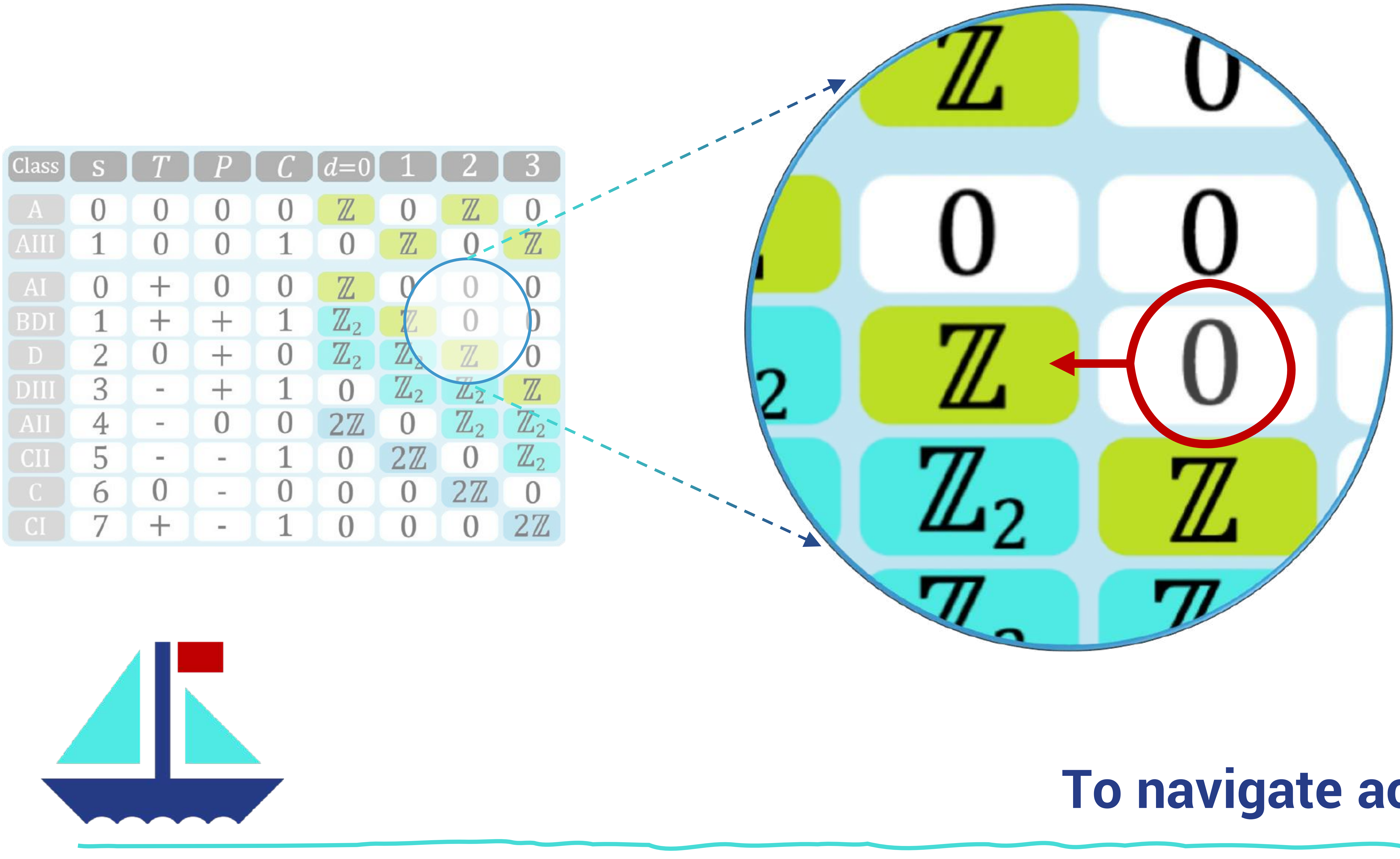
~~d~~ $\delta$: space dimension

Tenfold: 10 classes

| Class | $s$ | $T$ | $P$ | $C$ | $d=0$ | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| AIII | 1 | 0 | 0 | 1 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AI | 0 | + | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| BDI | 1 | + | + | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| D | 2 | 0 | + | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| DIII | 3 | - | + | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| AII | 4 | - | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| CII | 5 | - | - | 1 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| C | 6 | 0 | - | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 |
| CI | 7 | + | - | 1 | 0 | 0 | 0 | $2\mathbb{Z}$ |

Teo and Kane (2010)

# Defects in topological crystals:
# Creating Quantum Matter

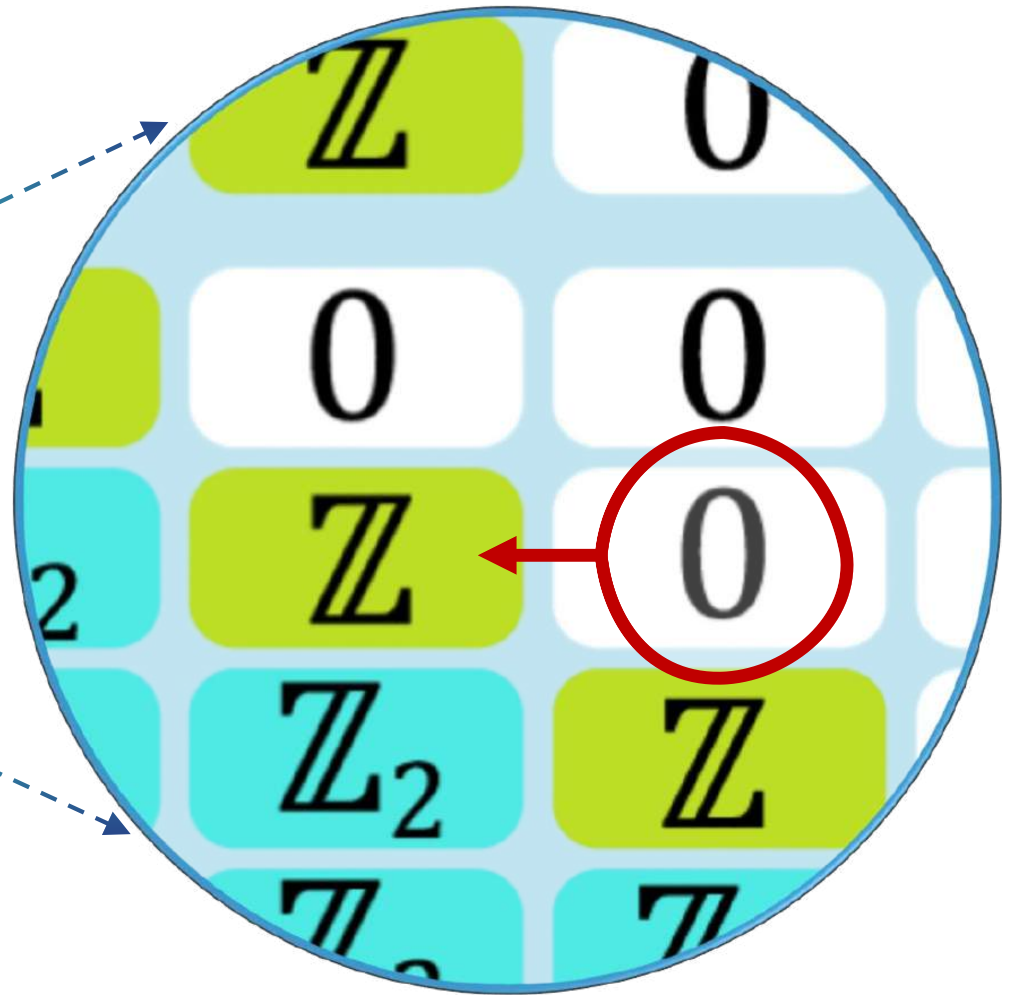


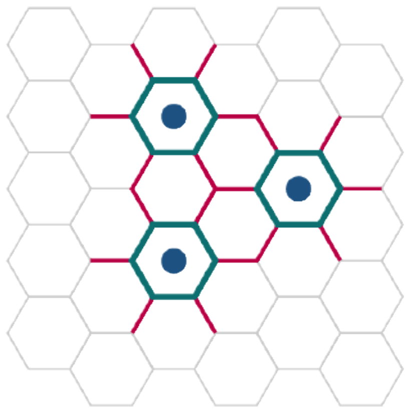
| Class | s | $T$ | $P$ | $C$ | $d=0$ | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| AIII | 1 | 0 | 0 | 1 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AI | 0 | + | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| BDI | 1 | + | + | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| D | 2 | 0 | + | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| DIII | 3 | - | + | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| AII | 4 | - | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| CII | 5 | - | - | 1 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| C | 6 | 0 | - | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 |
| CI | 7 | + | - | 1 | 0 | 0 | 0 | $2\mathbb{Z}$ |

To navigate across the tenfold classification

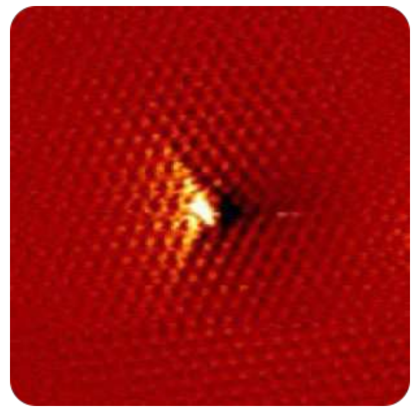
# Defects in topological crystals:
# Creating Quantum Matter

| Class | s | $T$ | $P$ | $C$ | $d=0$ | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| AIII | 1 | 0 | 0 | 1 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AI | 0 | + | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |
| BDI | 1 | + | + | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| D | 2 | 0 | + | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| DIII | 3 | - | + | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| AII | 4 | - | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| CII | 5 | - | - | 1 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| C | 6 | 0 | - | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 |
| CI | 7 | + | - | 1 | 0 | 0 | 0 | $2\mathbb{Z}$ |





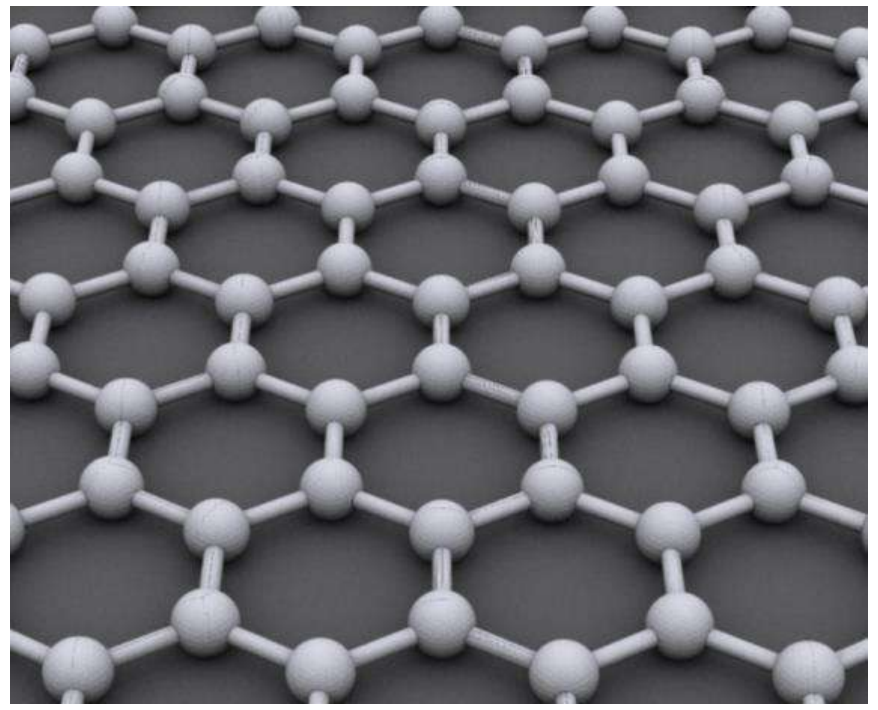
Kekule textures

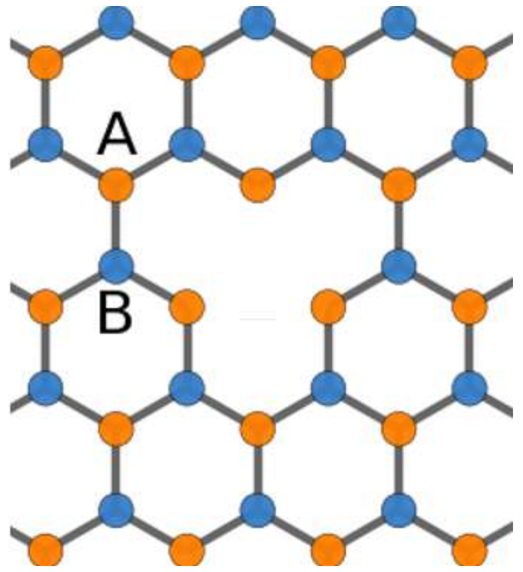


Lacunar Graphene

**Dig a hole in graphene makes it topological!**

Graphene





# The Symbol H(k,r): The Right Object

**The Wigner/Weyl symbol generalises the Bloch Hamiltonian to non-periodic systems.**

# Wigner Symbol

$$\mathcal{H}(\boldsymbol{k},\boldsymbol{r}) = \int_{-\infty}^{\infty} d\boldsymbol{r}' e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} \langle \boldsymbol{r} + \frac{\boldsymbol{r}'}{2} |\, H \,| \boldsymbol{r} - \frac{\boldsymbol{r}'}{2} \rangle$$

**Weyl transform**

$$\text{Hamiltonian} \rightarrow \text{Symbol}$$

$$H\left(\partial_{\boldsymbol{r}}, \hat{\boldsymbol{r}}\right) \rightarrow \mathcal{H}\left(\boldsymbol{k}, \boldsymbol{r}\right)$$

$$\partial_{\boldsymbol{r}} \rightarrow \boldsymbol{k}$$

$$\text{Operators} \rightarrow \text{Parameters}$$

# Wigner Symbol

Example

## Bloch Hamiltonian

With translation invariance

$$H = \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}^{\dagger} H(\boldsymbol{k}) c_{\boldsymbol{k}}$$

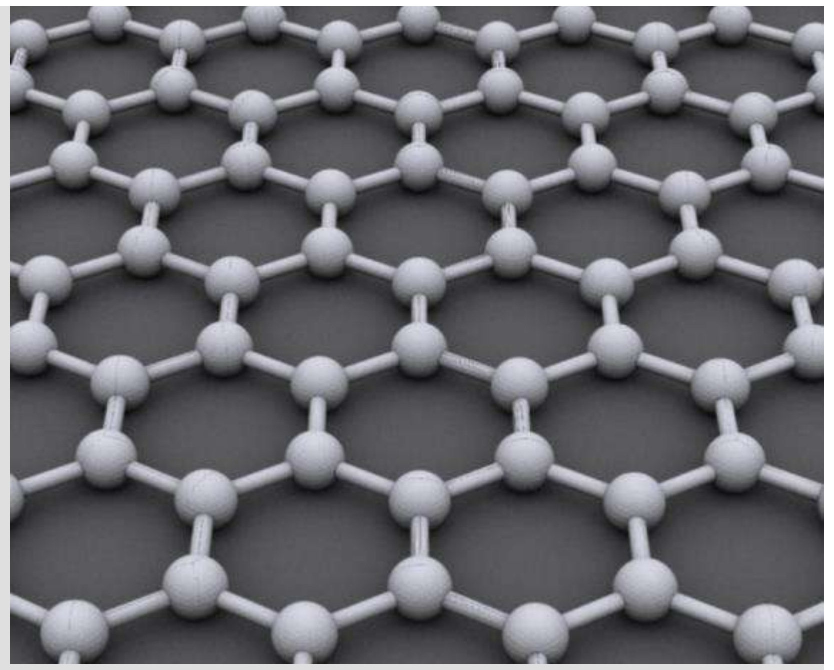
↓ Weyl transform

Symbol $$\mathcal{H}(\boldsymbol{k}) = H(\boldsymbol{k})$$ Bloch Hamiltonian

# Graphene

Example

## Symbol of Graphene



$$H_0 = -t \sum_i \sum_{\delta=0}^{2} a_i^\dagger b_{i+\delta} + h.c.$$

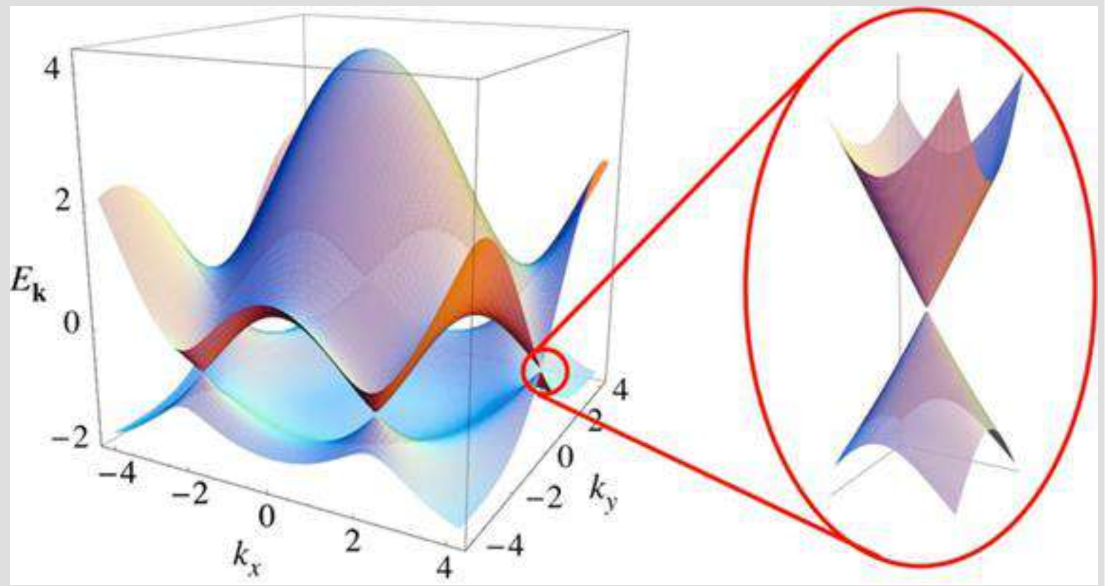
Translation invariance and low energy limit

**Symbol**
(Bloch Hamiltonian)

$$\mathcal{H}_0(\boldsymbol{k}) = k_x\, \sigma_x \otimes \sigma_z + k_y\, \sigma_y \otimes \mathbf{1}$$

Sublattice ↑    Valley ↑

$\sigma_i$ = Pauli matrices



Goft et al (2023)    Neto et al (2009)

# Topology Lives in the Symbol, Not in the Hamiltonian

## Hamiltonian

$$H = \sum_{i,j} t_{ij} c_i^\dagger c_j + \Delta\, c_i^\dagger c_j^\dagger + \Delta^\dagger c_i c_j$$

Naive: cannot apply
tenfold classification

**✗ No spectral gap**

## Symbol

$$\mathcal{H}(\boldsymbol{k},\boldsymbol{r}) = \boldsymbol{h}_a \cdot \boldsymbol{\gamma}_a + \boldsymbol{h}_s \cdot \boldsymbol{\gamma}_s \equiv \boldsymbol{h}(\boldsymbol{k},\boldsymbol{r}) \cdot \boldsymbol{\gamma}$$

p components

q+1 components

Anti commuting
Dirac matrices

**✓ Spectral gap exists**

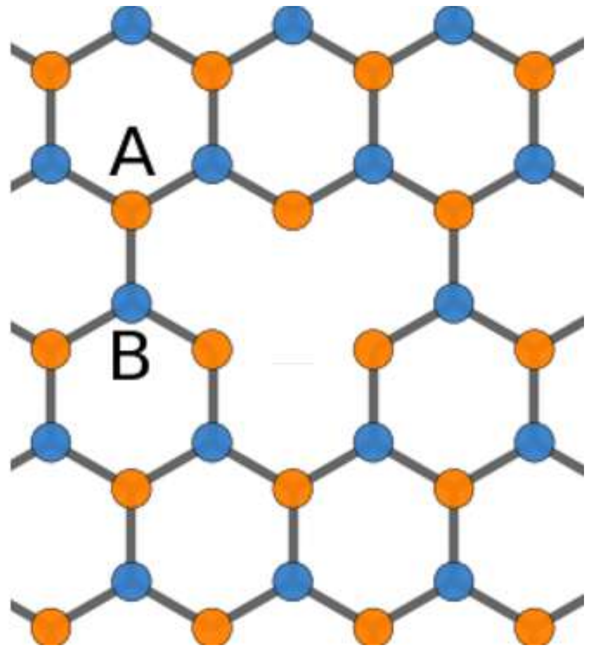
**Topology is encoded in the symbol, not the Hamiltonian. This distinction is invisible when translation symmetry holds.**

# Wigner Symbol

1. **No translation invariance** → **Use the Symbol**
2. **Hamiltonian spectrum** $\neq$ **Symbol spectrum**
3. **What about topology?**

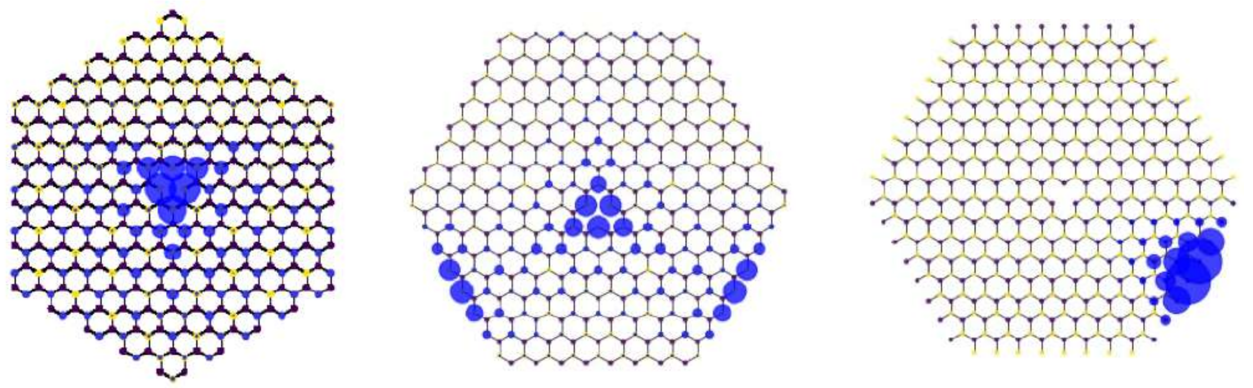
   **In math**: Atiyah-Singer index theorem
   **In physics**: Bulk-Edge correspondence

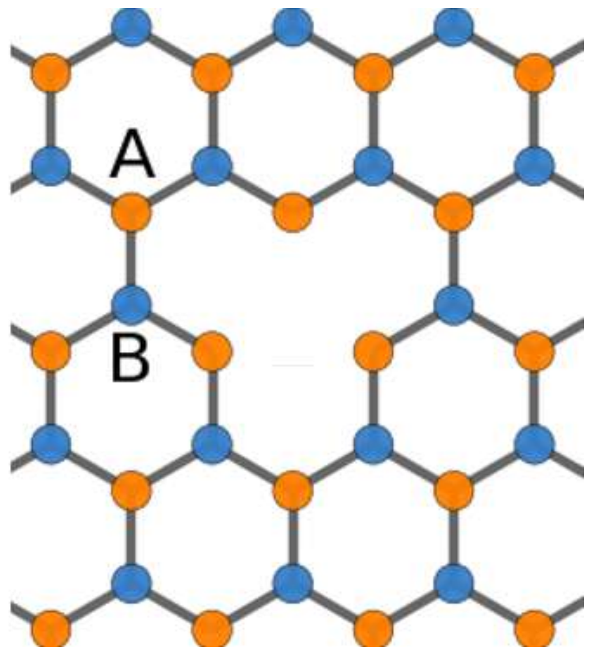
# Zero modes count topology - and dislocations make it visible



# Zero modes count topology - and dislocations make it visible

## Zero modes - Edge states



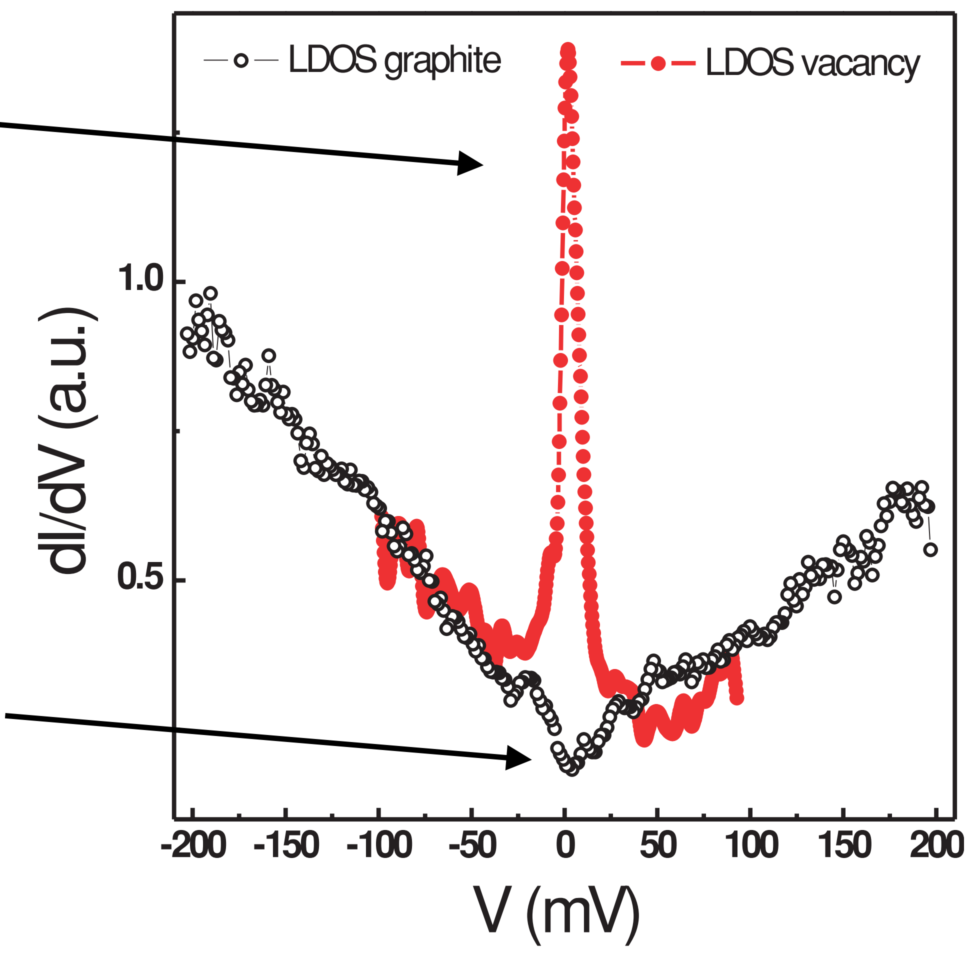



Dig a hole in graphene makes it topological!

**Topological ?**

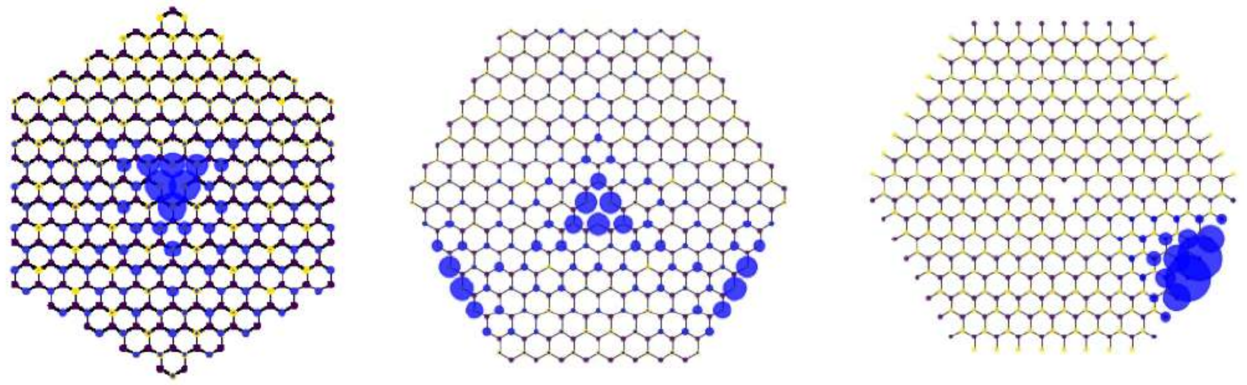
Single vacancy
Zero mode

No vacancy

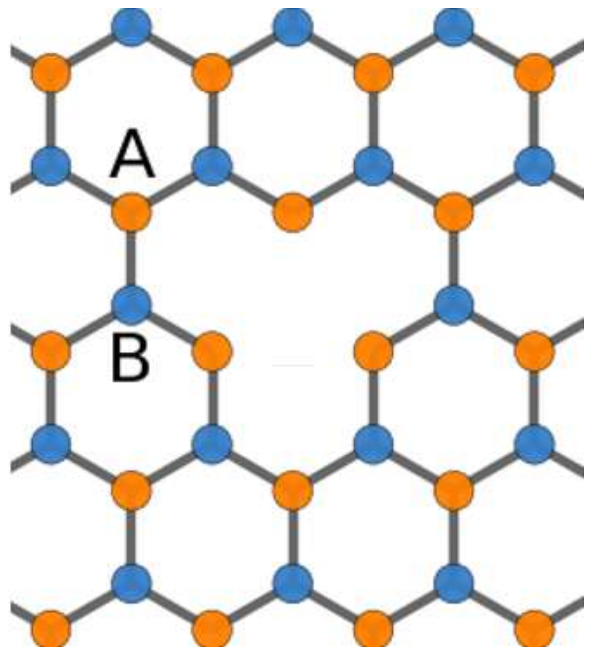


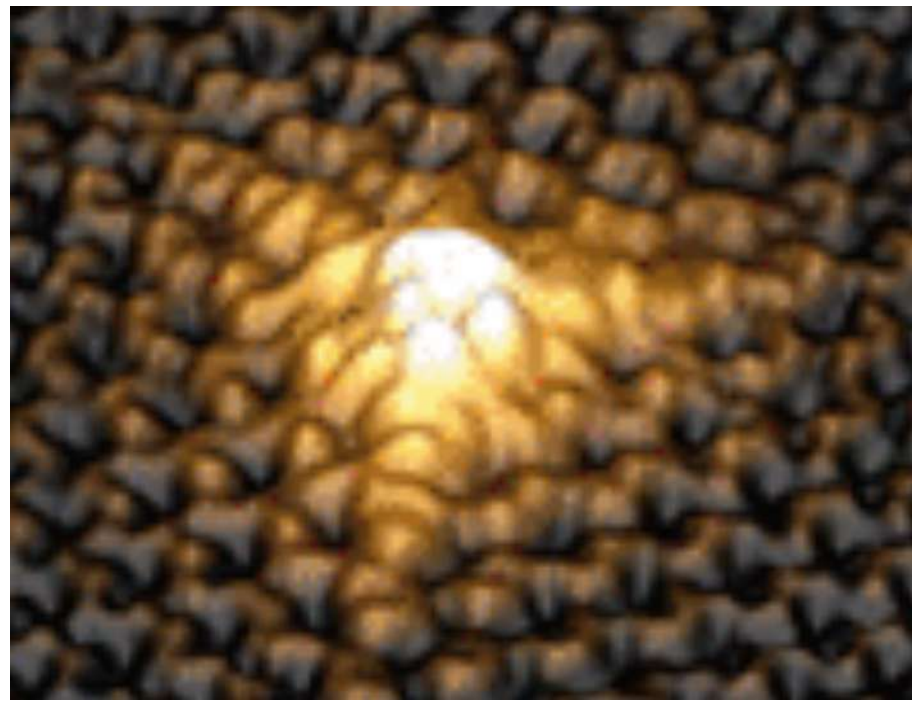
# Zero modes count topology - and dislocations make it visible

## Zero modes - Edge states

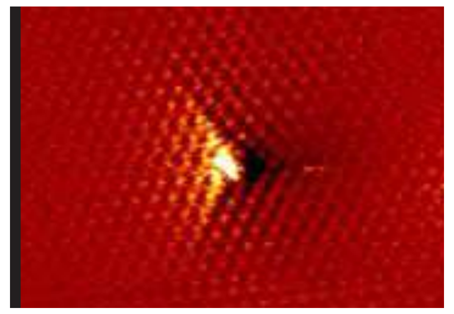


Dig a hole in graphene makes it topological!



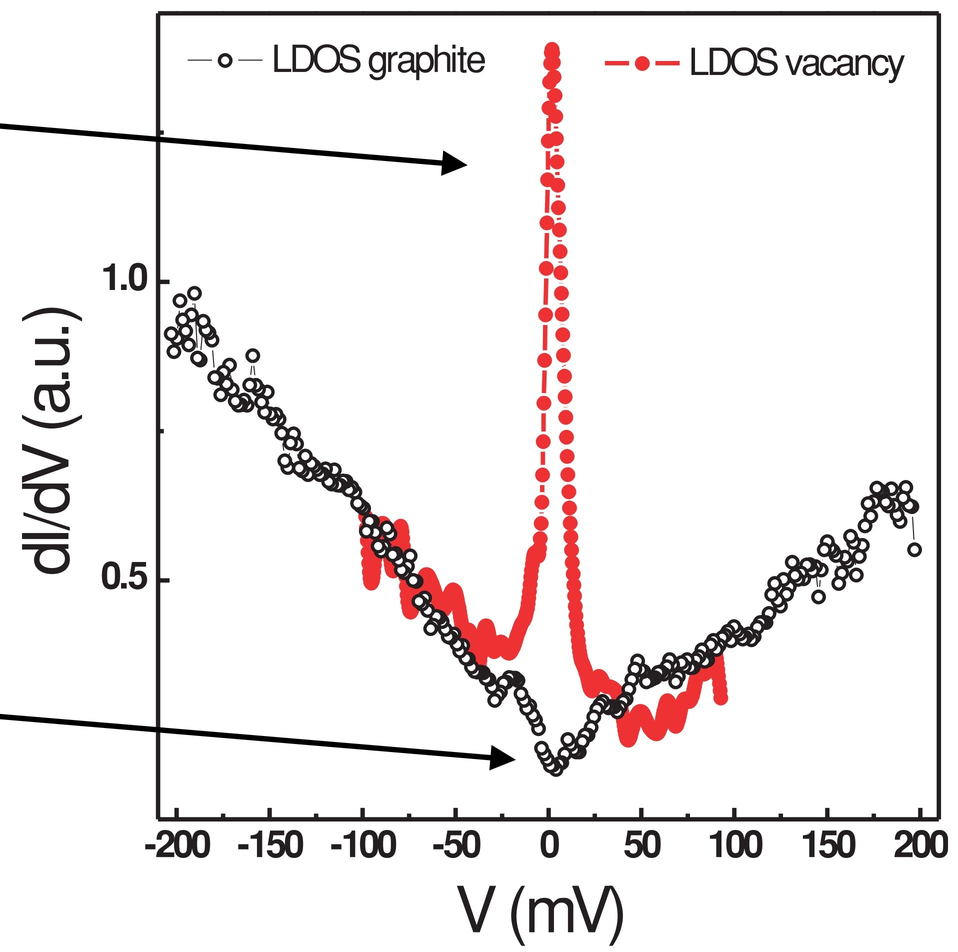



Ugeda, et al (2010) Phys. Rev. Lett.



**Topological ?**

Single vacancy
Zero mode



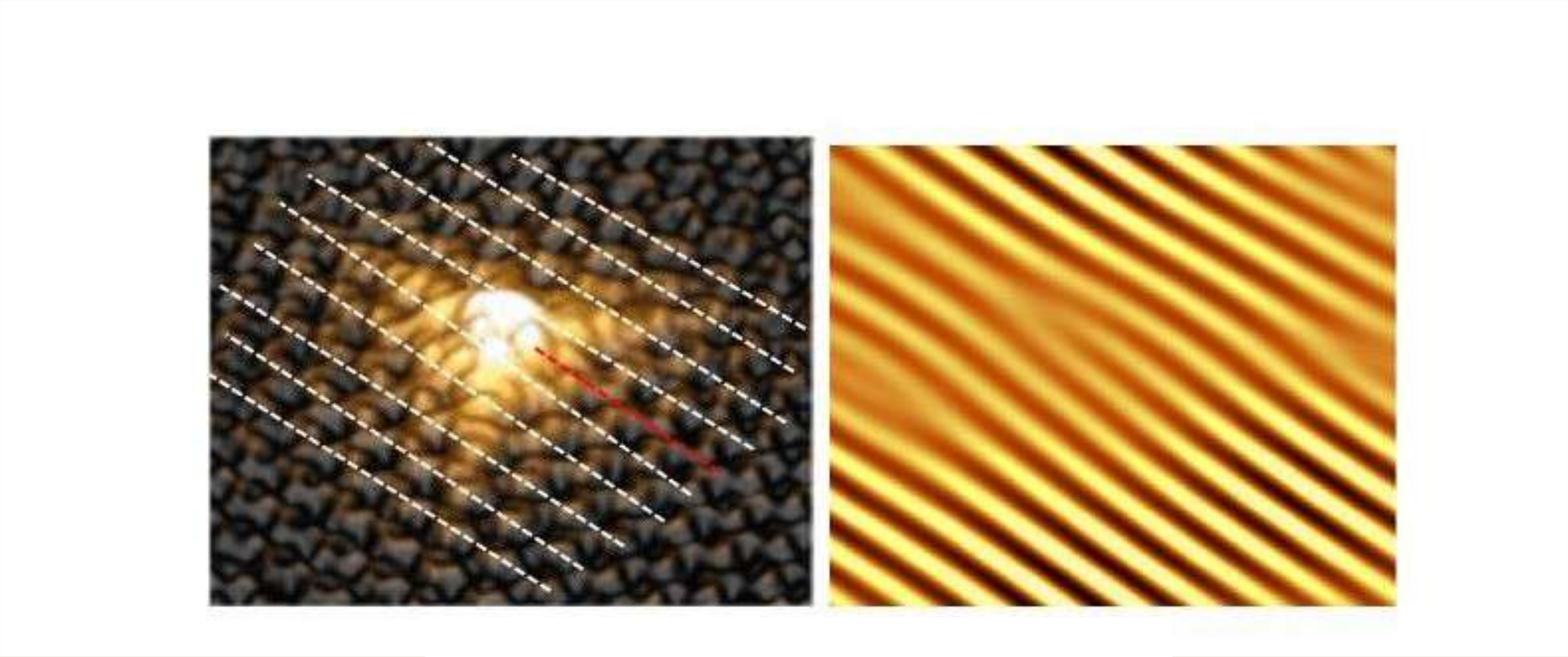

No vacancy

$$\rho(\boldsymbol{r}) = -\frac{1}{\pi}\int dE\,\mathrm{Im}\sum_n \frac{|\varphi_n(\boldsymbol{r})|^2}{E - E_n + i0^+},$$

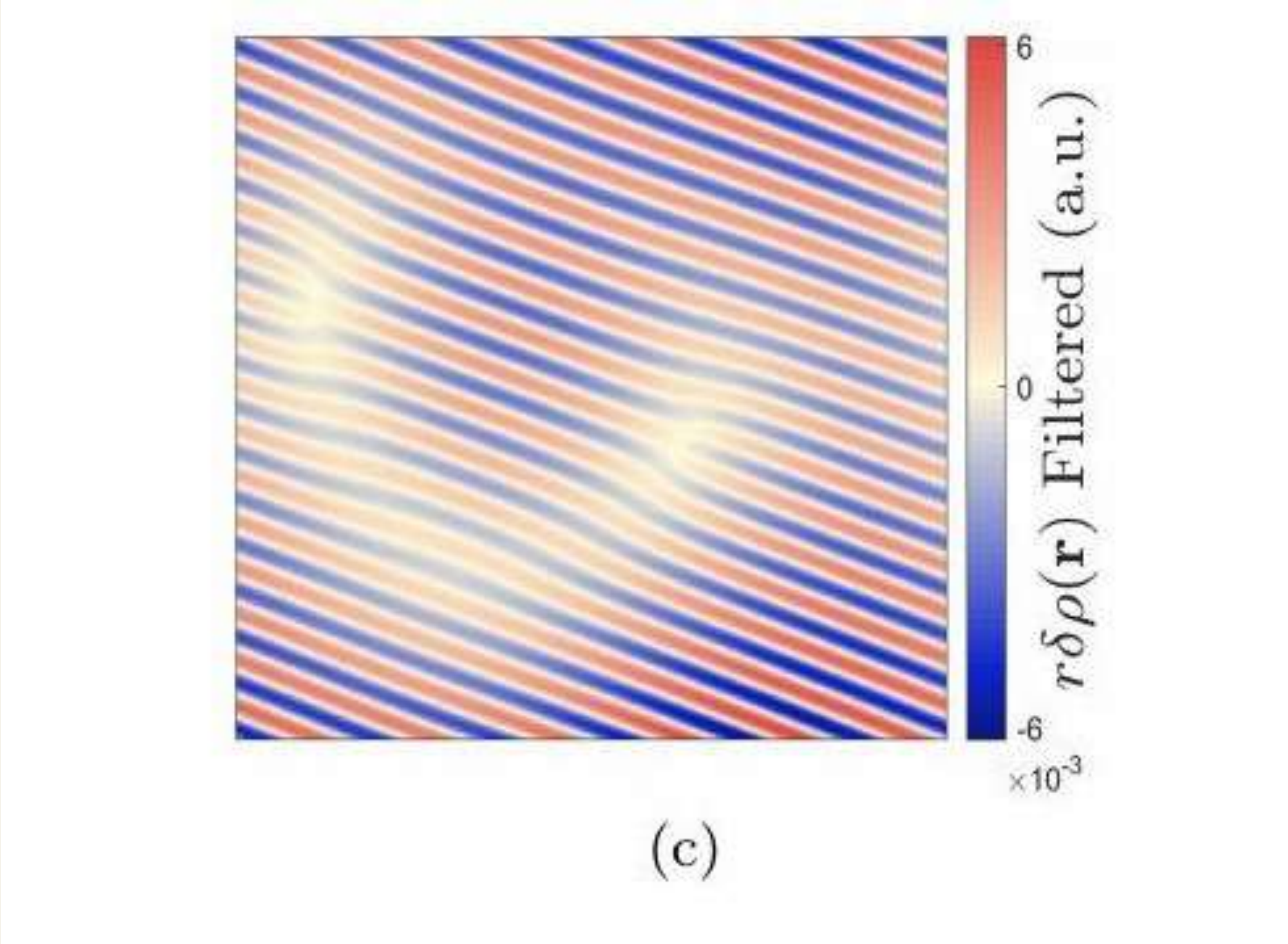
Local charge density

# Made experimental — STM data on a graphene vacancy

*Y. Abulafia & E. Akkermans, PRL submitted (2026)*



*(a) Raw STM image of a graphene monovacancy (Ugeda et al., 2010) with wavefront guides — one wavefront terminates at the defect. (b) After the same Fourier filter applied to tight-binding numerics: a single dislocation, ν = 1.*

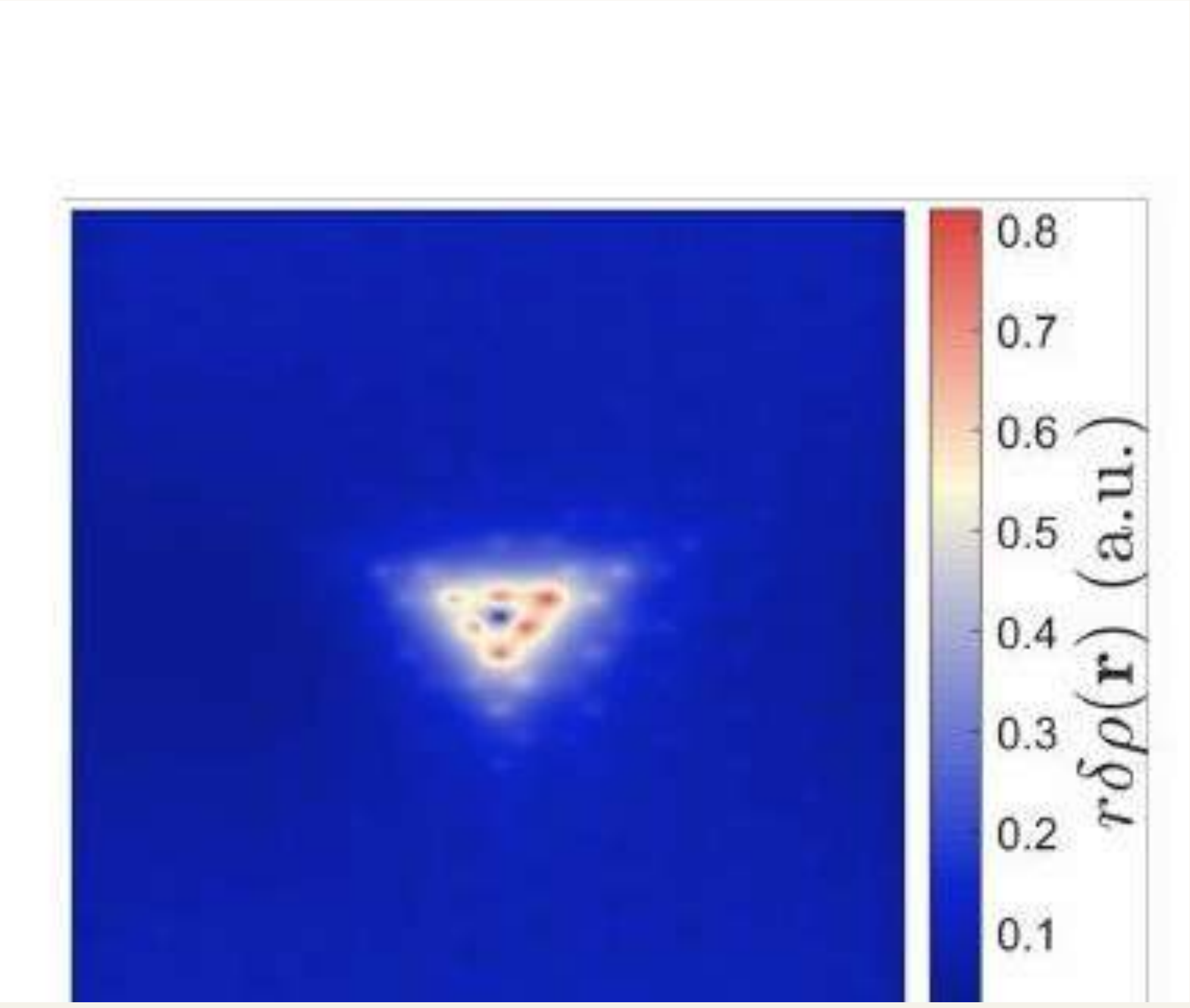



*Numerical tight-binding rδρ(r) after the identical protocol — same dislocation, same topology.*

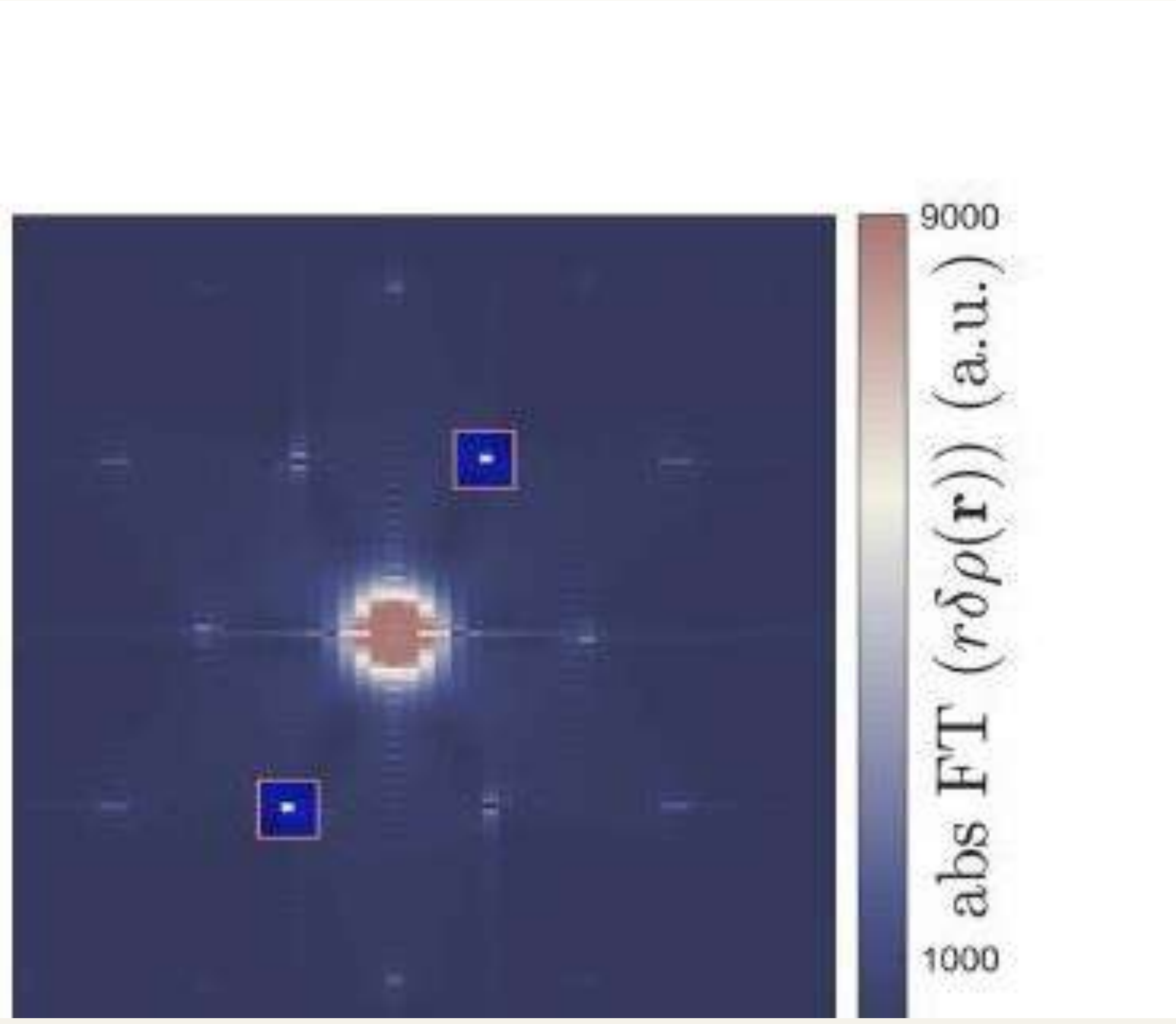
**Existing STM data already contains the topological information.**

*No new instrumentation needed — the dislocation is recovered by re-analysis of published images.  Next slide: how.*

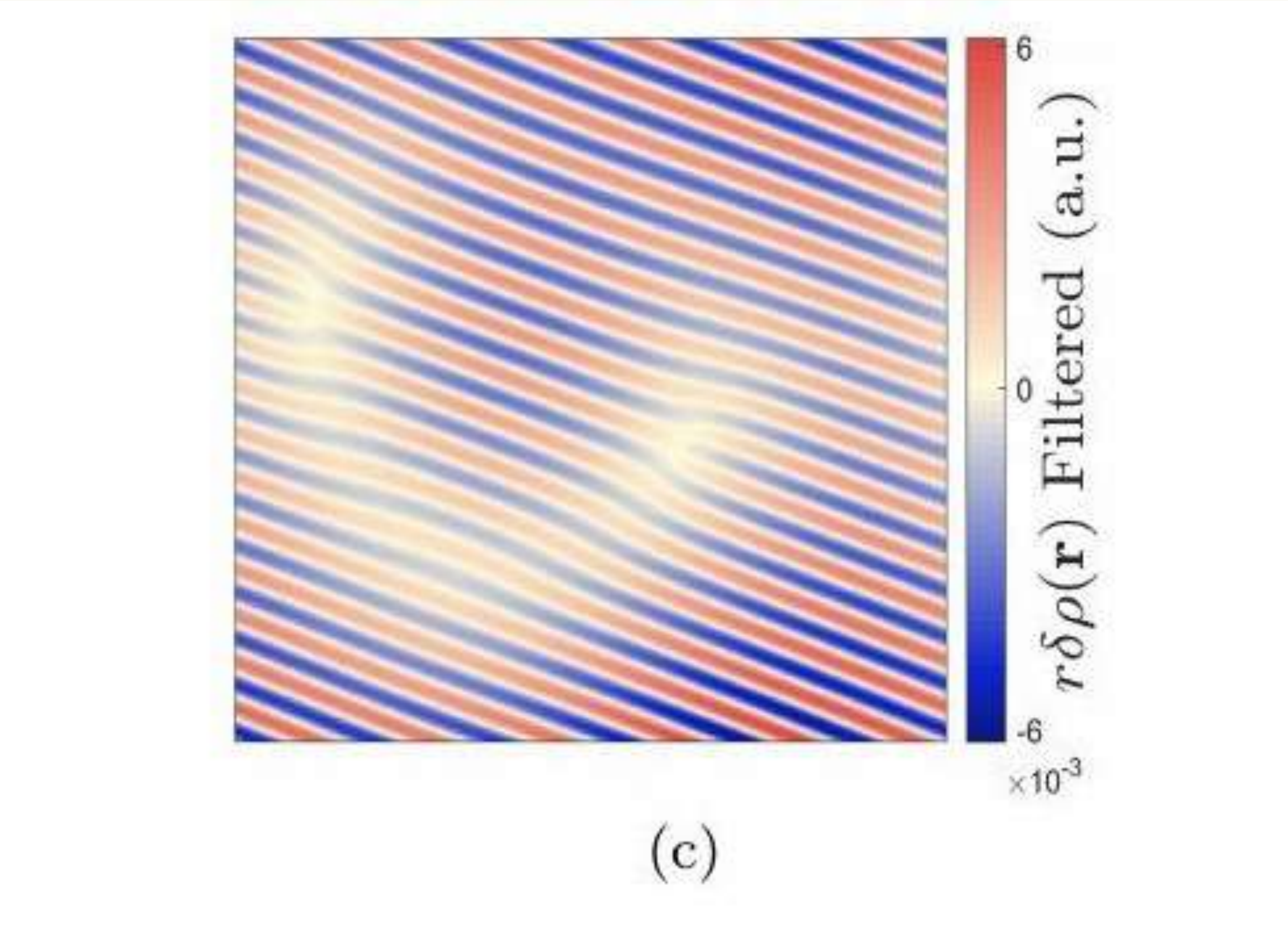
# The filtering protocol — how the dislocation is revealed




FT




filter + IFT




**(a) r · δρ(r)**

*input — tight-binding or STM*

**(b) | FT( r · δρ ) |**

*six satellite peaks — pick one pair*

**(c) filtered r · δρ**

*a single wavefront dislocation, ν = 1*

① **Pre-weight: form r · δρ(r).**
*δρ has a 1/r tail; multiplying by r flattens it, concentrating Fourier weight into sharp peaks rather than diffuse rings.*

② **Fourier-transform.**
*| FT(r·δρ) | shows six satellite peaks three intervalley pairs at $\pm\Delta K_n$ related by $C_3$ symmetry. This sextet is fermion doubling itself.*

③ **Filter — keep one diametrically opposite pair, complex-valued.**
*Set all other components (including DC) to zero. Crucially retain BOTH amplitude AND phase within the window.*

④ **Inverse-transform.**
*The filtered density isolates the intervalley interference. The phase winding of the zero-mode wavefunction prints as a dislocation in the wavefronts.*

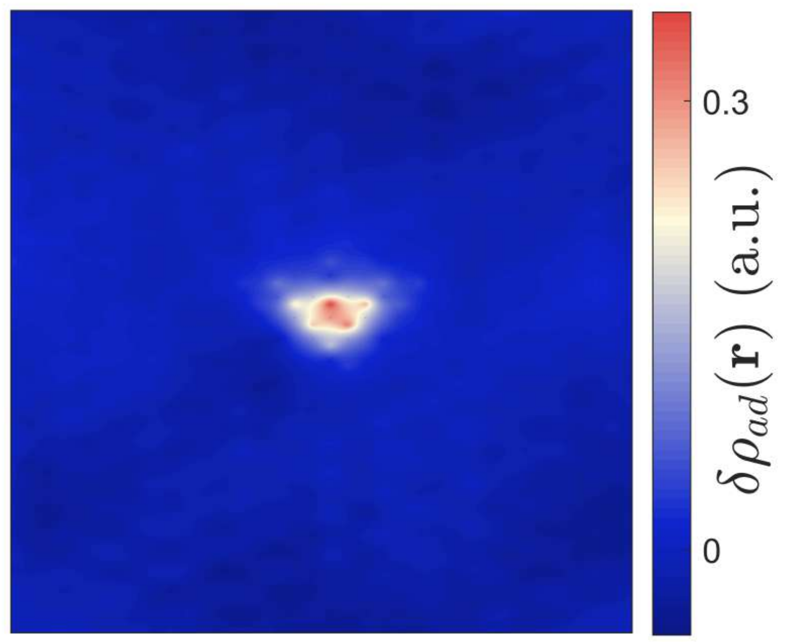
⑤ **The same window discriminates automatically.**
*Adatom: |FT| acquires Friedel-oscillation rings at $2k_F$ around the Dirac points (broken chiral symmetry). The two-peak window excludes them no dislocation in the IFT.*

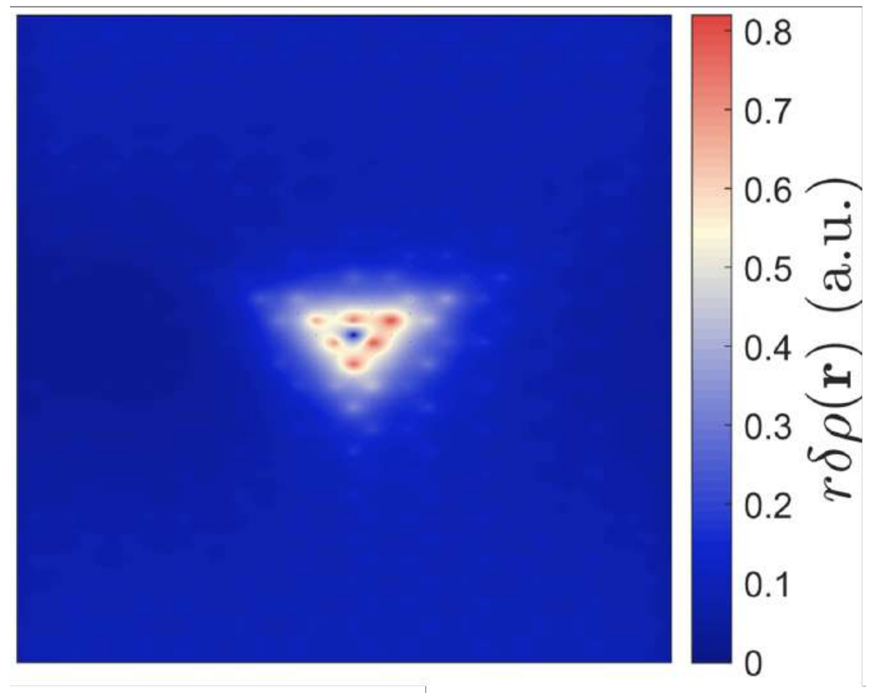
***Identical protocol on STM data and numerics.***

# Quantitative: contour-independent winding number

$$\rho(\boldsymbol{r}) = -\frac{1}{\pi}\int dE\,\mathrm{Im}\sum_n \frac{|\varphi_n(\boldsymbol{r})|^2}{E - E_n + i0^+},$$

$$\rho(\boldsymbol{r}) = -\mathrm{Re}\Big(iF(r)\sin\theta\, e^{i\chi(\boldsymbol{r})}\Big)$$




## Direct measurement

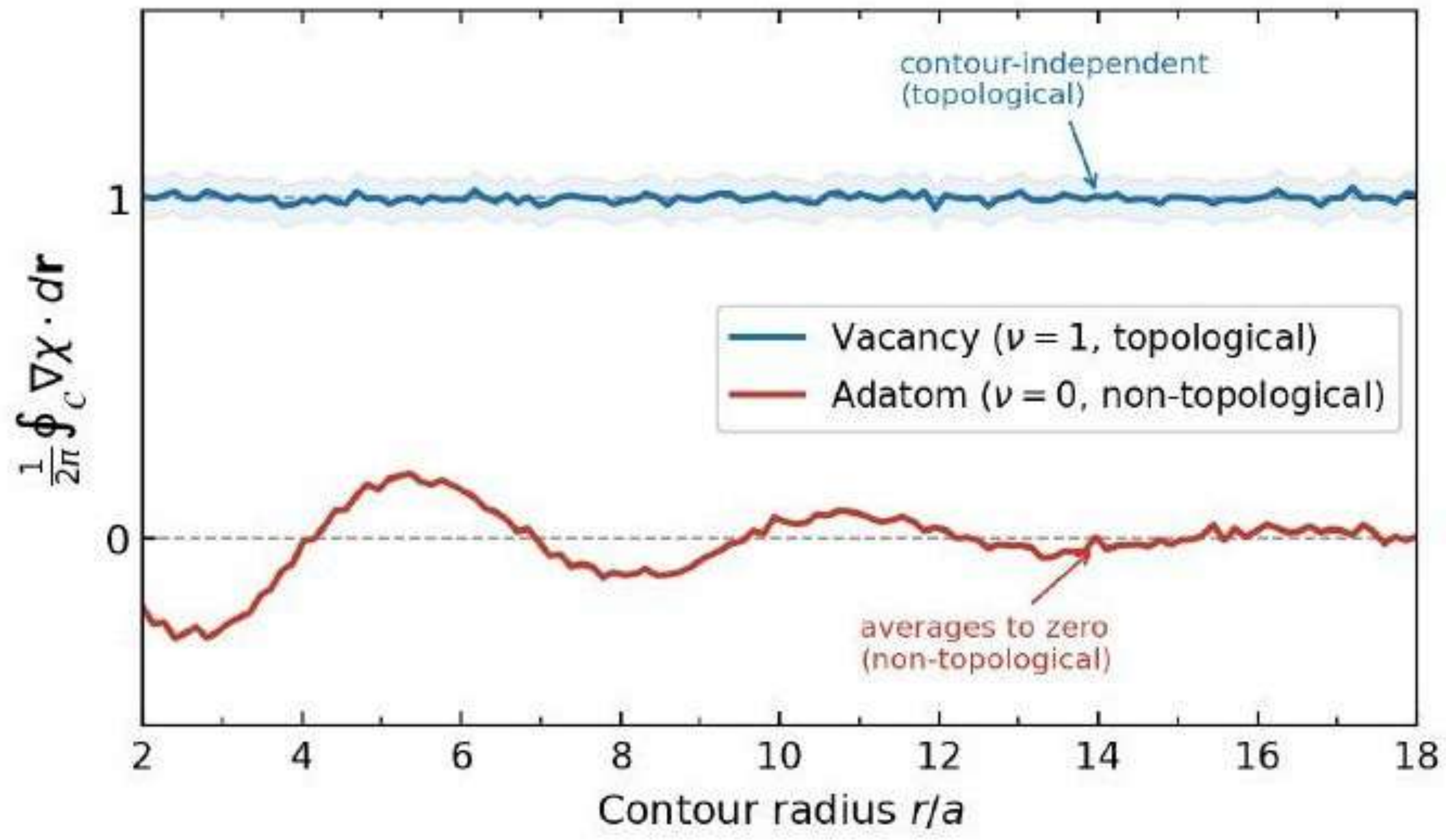
Integrate ∇χ around contours of growing radius:

$$\frac{1}{2\pi}\oint_{\mathcal{C}} \nabla\chi(\boldsymbol{r})\cdot d\boldsymbol{r} = \nu \in \mathbb{Z},$$

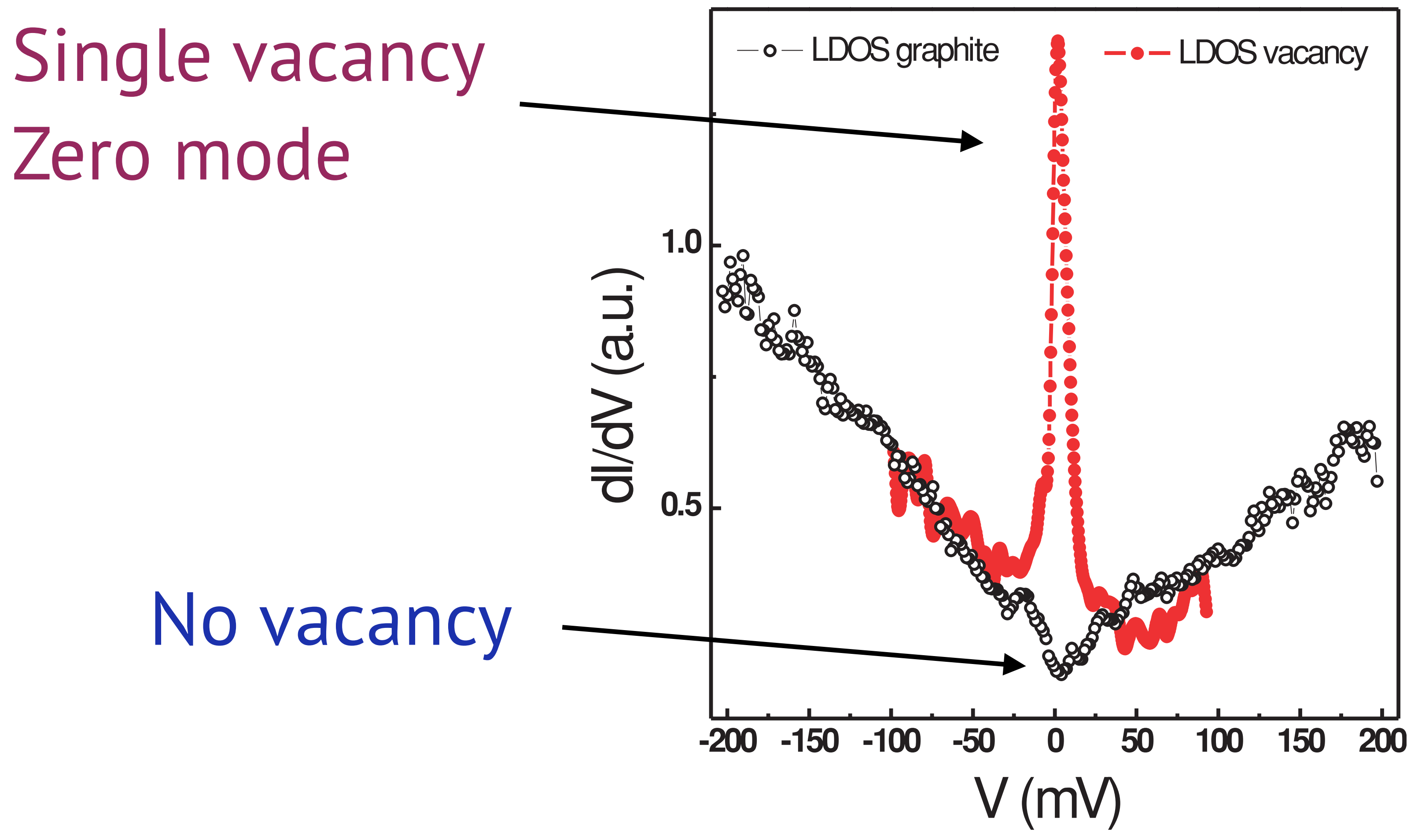
Vacancy (blue): ν = 1 — flat across all radii ⇒ genuine topological invariant.

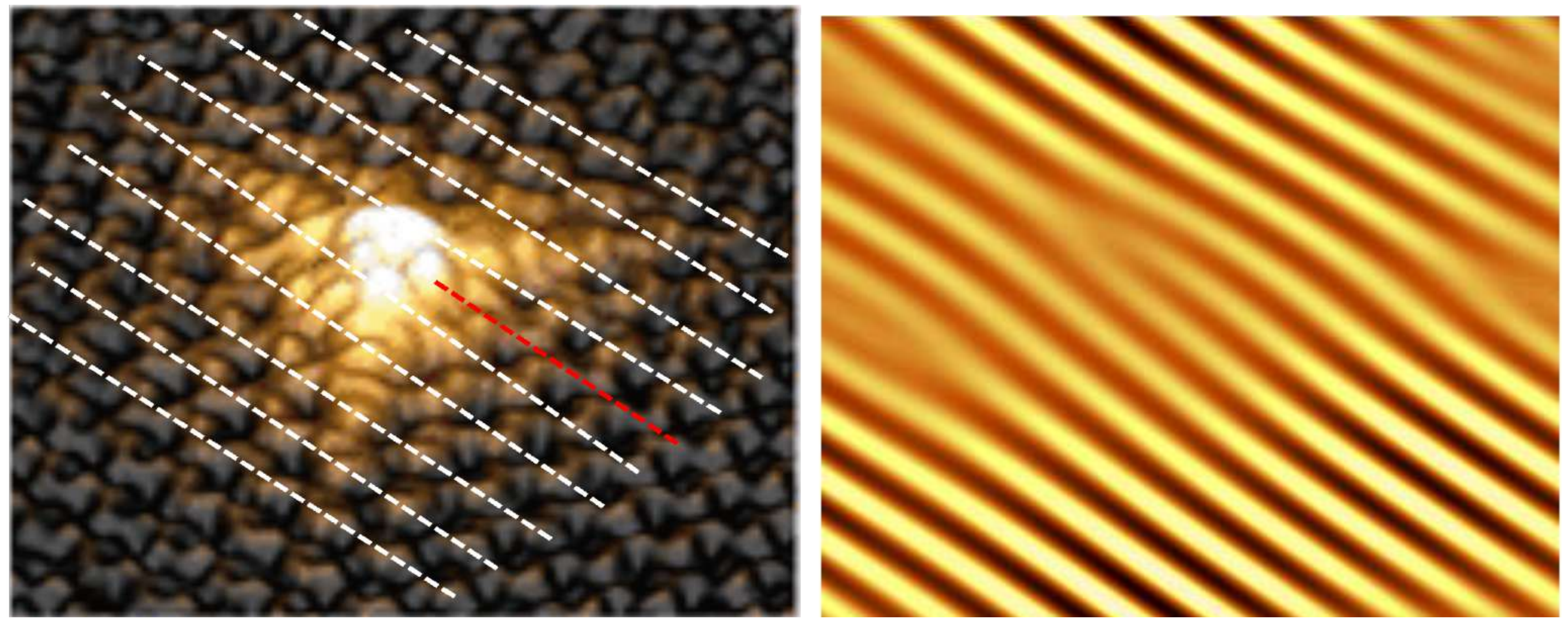
Adatom (red): winding wanders and averages to zero — Friedel oscillations confine it locally.

***Same protocol, opposite verdict — discriminating, not just detecting, topology.***



# Zero modes count topology - and dislocations make it visible

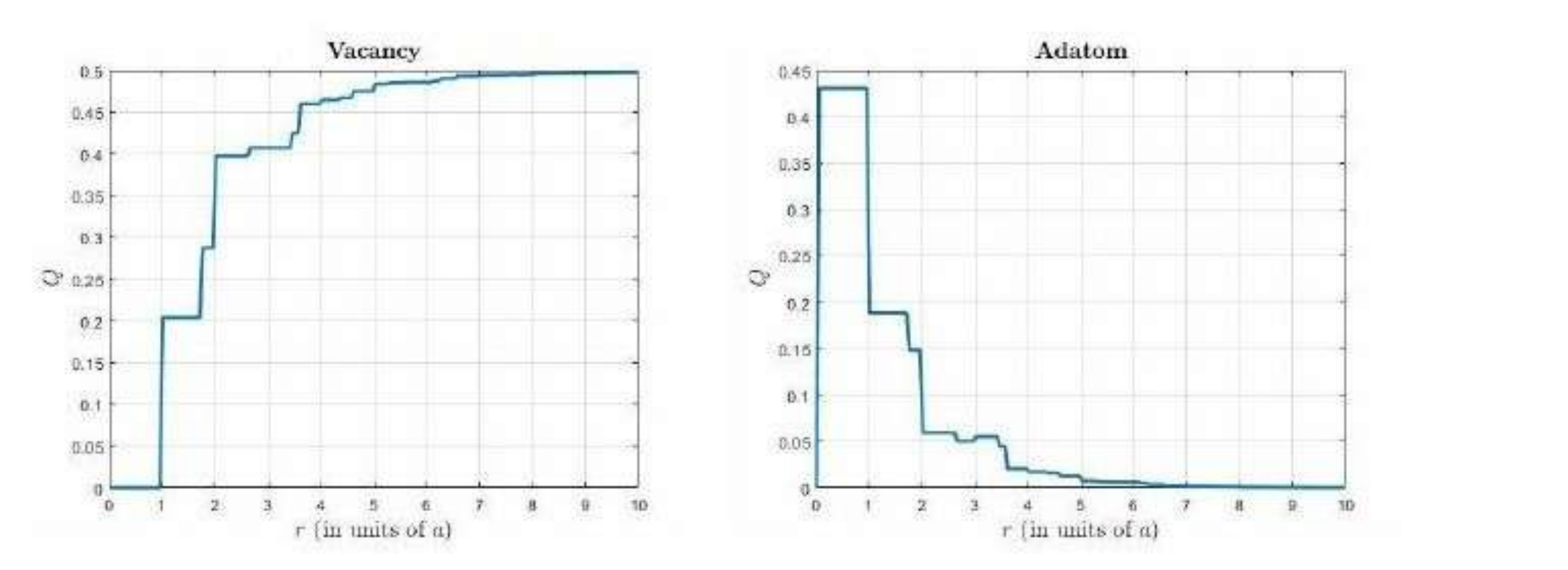


$$N_{\mathrm{zm}} = |\mathrm{Index}\,\mathcal{D}| = |\nu|.$$

# Two more measurable signatures

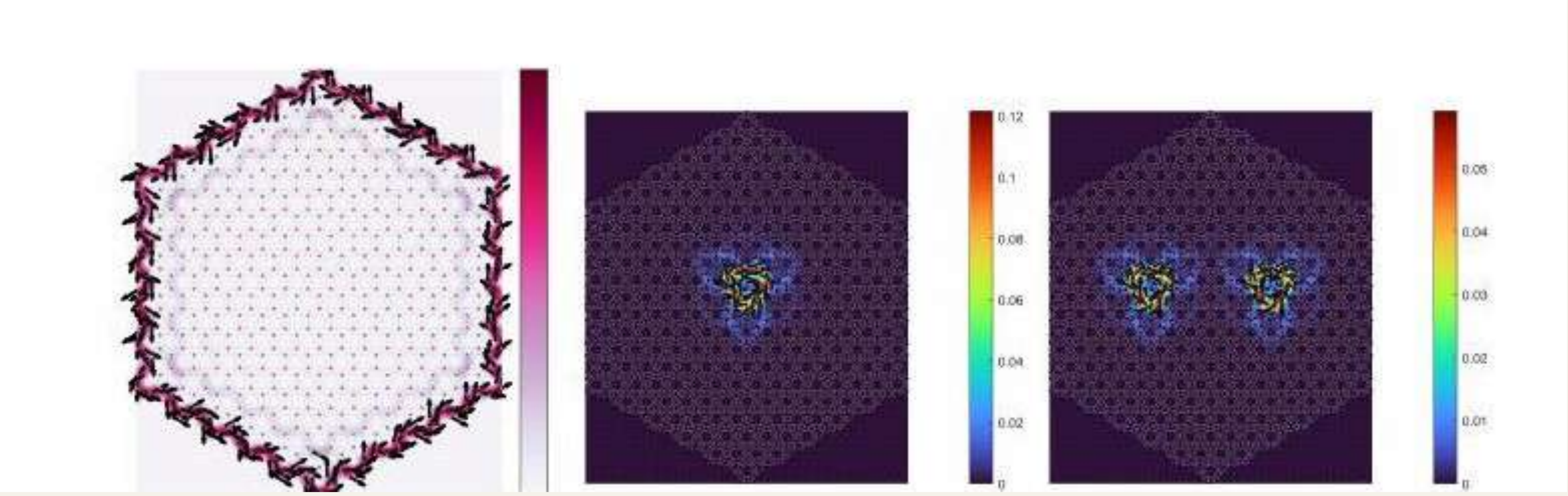
## Fractional charge




**$Q = -e/2 \cdot \nu$**

Integrated δρ around the vacancy saturates at e/2; adatom: 0. Direct, gauge-invariant readout of the $Z_2$ invariant.

## Chiral current reversal



**Edge current ⇄ vacancy-mode current**

Vacancy zero mode circulates opposite to the chiral edge — exact analogue of a vortex in a p-wave superconductor (Majorana physics in disguise).

**Three independent signatures, one invariant.** *Dislocations · e/2 charge · current reversal → all agree on ν = C·m mod 2.*

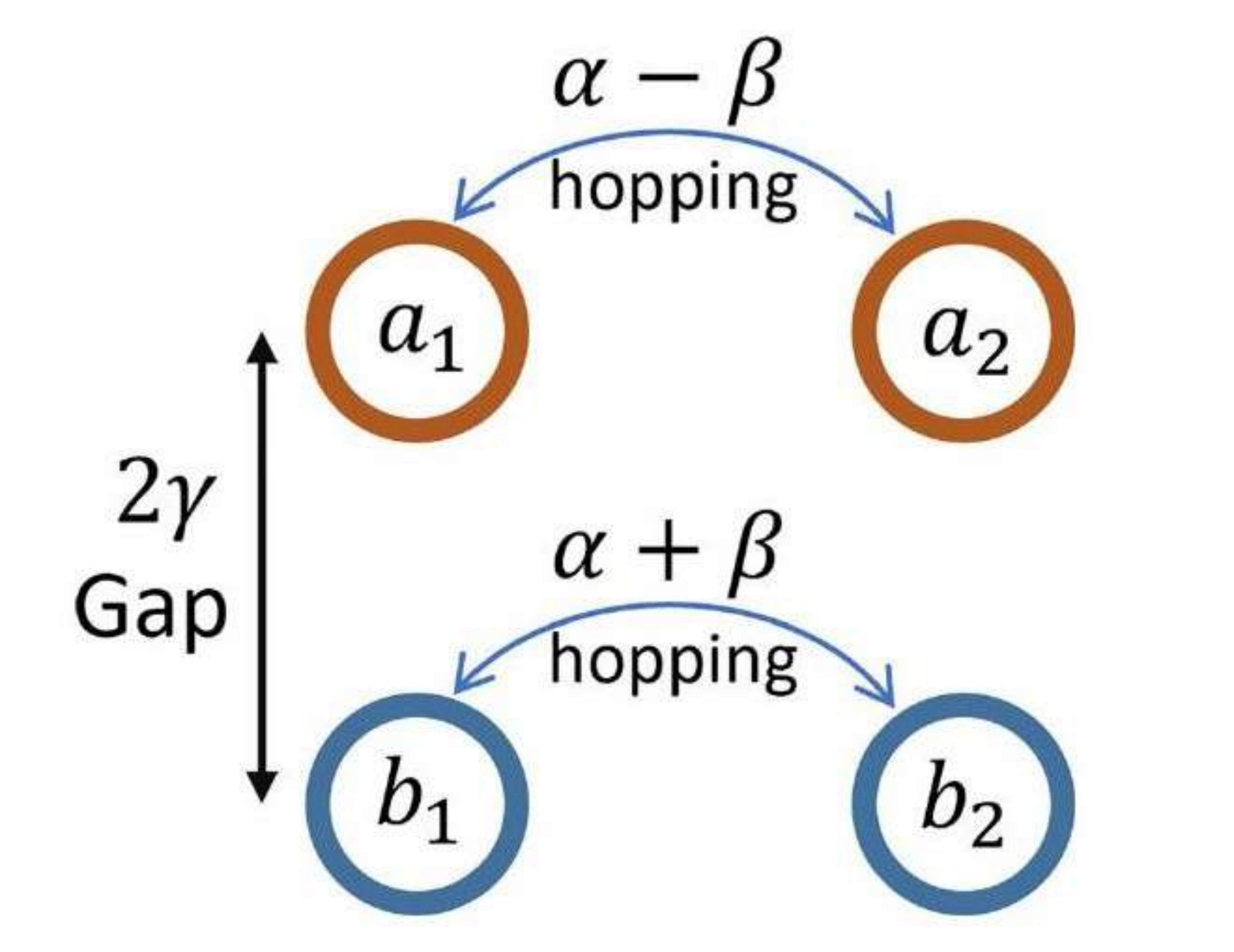
# Two-Qubit Entanglement is Topological

*Orion & Akkermans — PRB 111, 245408 (2025)*

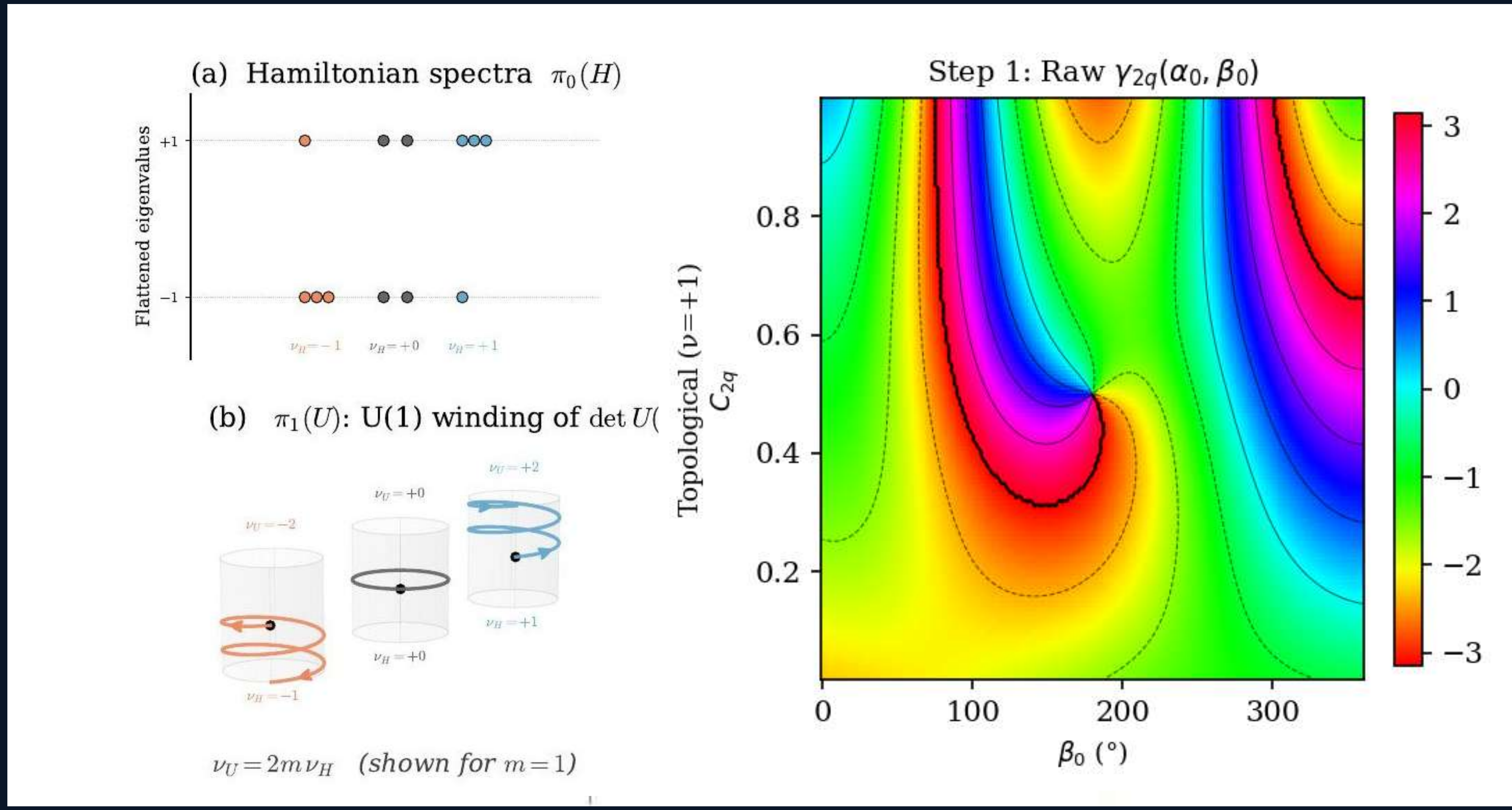
- The Wootters concurrence operator equals the time-reversal operator — not a coincidence
- The space of two-qubit Hamiltonians has five disconnected topological phases ($\mathbb{Z}_5$)
- Bell states are topological zero modes at phase boundaries

# The same framework classifies two qubits

*Tenfold classification → an entangling two-qubit Hamiltonian is a 4-site quantum graph*




*Time-reversal-invariant $H_2q$ maps to a 4-site chiral lattice with two hopping pairs.*




*Energy crossings vs. γ define five topological sectors $\pi_0(R_0(4)) = \mathbb{Z}_5$ ; zero modes are Bell states.*

***Same chiral structure · same index theorem · same Z / $Z_2$ invariants — now the "lattice sites" are computational basis states.***

# Concurrence describes entanglement

Consider two distinguishable qubits described by $|\psi\rangle$

$$C_\psi \equiv |\langle\psi|\Theta|\psi\rangle| \in [0,1]$$

Anti-unitary time reversal

$$\Theta = (\sigma_y \otimes \sigma_y)K$$

$K$: Complex conjugate operator

$\sigma_y$: $2 \times 2$ Pauli matrix

Product states

$$|00\rangle \to C = 0$$

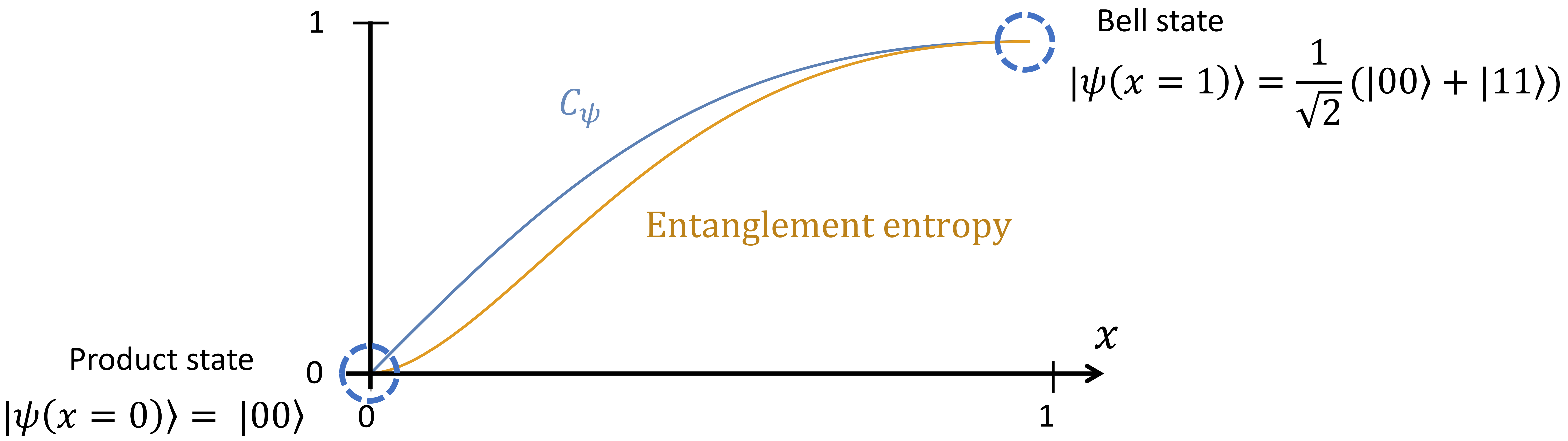
Bell states

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \to C = 1$$

Wootters (1998)

# Concurrence describes entanglement

For the intermediate state $|\psi(x)\rangle = \frac{1}{\sqrt{1+x^2}}(|00\rangle + x|11\rangle)$




$C_\psi$ and entanglement entropy are equivalent

Wootters (1998)

# Wootters Concurrence and Time-Reversal Symmetry

**Can the tenfold classification — built for large systems — say anything about a 4-dimensional Hilbert space?**

**Wootters concurrence:** $C_\psi \equiv |\langle\psi|\Theta|\psi\rangle| \in [0,1] \qquad \Theta = (\sigma_y \otimes \sigma_y)K$

**$\Theta = (\sigma_y \otimes \sigma_y)K$:** Time-reversal operator for two spin-1/2 particles. K = complex conjugation in computational basis.

**$\Theta 2 = +1_4$:** Squares to +1 → class AI symmetry. This is the same operator that classifies the symmetry class.

**$C\psi = 0$:** Separable state. $C\psi = 1$: maximally entangled (Bell) state. Same Θ defines both entanglement and topology.

**Not a coincidence:** The operator that measures entanglement identifies the symmetry class. This is the heart of the connection.

# Cartan decomposition

Two qubit Hamiltonians: $g = \sigma_i \otimes I,\ I \otimes \sigma_j,\ \sigma_i \otimes \sigma_j$

Cartan
Decomposition:
$g = h \oplus m$

$$\Theta = (\sigma_y \otimes \sigma_y)K$$

$$C_\psi \equiv |\langle \psi | \Theta | \psi \rangle|$$

Orion and Akkermans (2025)

Khaneja et al (2001),
Zhang et al (2003)

# Cartan decomposition

Two qubit Hamiltonians: $g = \sigma_i \otimes I,\ I \otimes \sigma_j,\ \sigma_i \otimes \sigma_j$

Cartan Decomposition:
$g = h \oplus m$

$\Theta = (\sigma_y \otimes \sigma_y)K$

$m = \sigma_i \otimes \sigma_j$

**Entangling!**

$\Theta m \Theta^{-1} = m$

$C_\psi \equiv |\langle\psi|\Theta|\psi\rangle|$

Orion and Akkermans (2025)

Khaneja et al (2001),
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# Cartan decomposition

Two qubit Hamiltonians: $g = \sigma_i \otimes I,\ I \otimes \sigma_j,\ \sigma_i \otimes \sigma_j$

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$\Theta m \Theta^{-1} = m$

$C_\psi \equiv |\langle\psi|\Theta|\psi\rangle|$

$h = \sigma_i \otimes I, I \otimes \sigma_j$

Non-entangling

$\Theta h \Theta^{-1} = -h$

Orion and Akkermans (2025)

Khaneja et al (2001),
Zhang et al (2003)

# Cartan decomposition

Two qubit Hamiltonians: $g = \sigma_i \otimes I,\ I \otimes \sigma_j,\ \sigma_i \otimes \sigma_j$

Cartan Decomposition:

$g = h \oplus m$

$\Theta = (\sigma_y \otimes \sigma_y)K$

$C_\psi \equiv |\langle\psi|\Theta|\psi\rangle|$

$m = \sigma_i \otimes \sigma_j$

**Entangling!**

$\Theta m \Theta^{-1} = m$

Class AI

$h = \sigma_i \otimes I, I \otimes \sigma_j$

Non-entangling

$\Theta h \Theta^{-1} = -h$

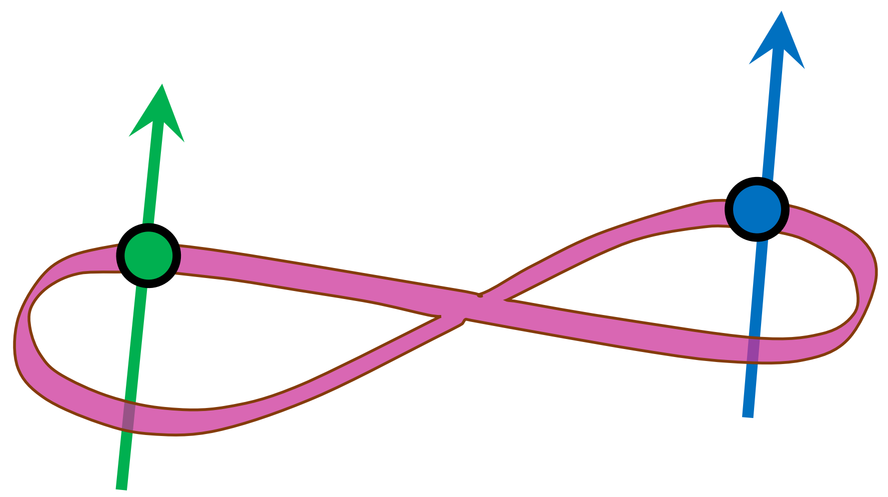
Class A

Orion and Akkermans (2025)

Khaneja et al (2001), Zhang et al (2003)

# Two qubits in the tenfold classification

| Class | $\Theta$ | $\Pi$ | $S$ | $d=0$ |
|---|---|---|---|---|
| A | 0 | 0 | 0 | $\mathbb{Z}$ |
| AIII | 0 | 0 | 1 | 0 |
| AI | + | 0 | 0 | $\mathbb{Z}$ |
| BDI | + | + | 1 | $\mathbb{Z}_2$ |
| D | 0 | + | 0 | $\mathbb{Z}_2$ |
| DIII | − | + | 1 | 0 |
| AII | − | 0 | 0 | $2\mathbb{Z}$ |
| CII | − | − | 1 | 0 |
| C | 0 | − | 0 | 0 |
| CI | + | − | 1 | 0 |



$d = 0$

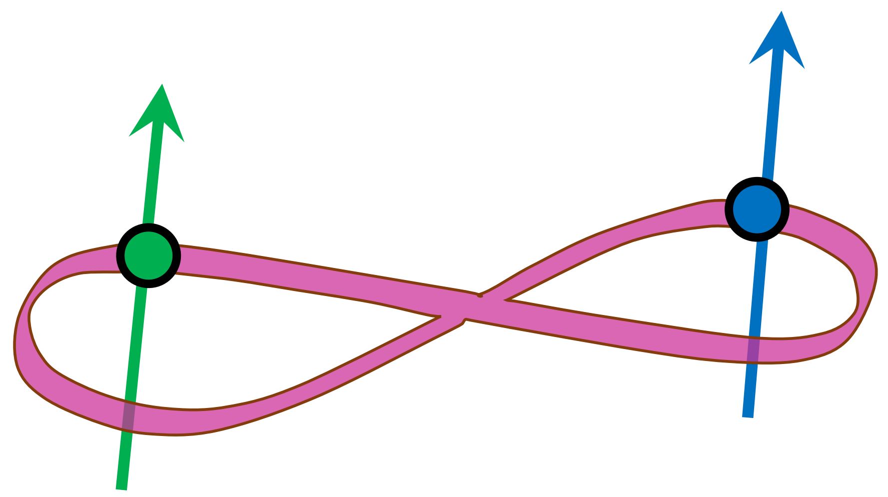
$\Theta = (\sigma_y \otimes \sigma_y)K$ $\qquad$ $\Theta^2 = +$

No $\Pi$ $\qquad$ $\Pi^2 = "0"$

# Two qubits in the tenfold classification

| Class | $\Theta$ | $\Pi$ | $S$ | $d = 0$ |
|---|---|---|---|---|
| A | 0 | 0 | 0 | $\mathbb{Z}$ |
| AIII | 0 | 0 | 1 | 0 |
| AI | + | 0 | 0 | $\mathbb{Z}$ |
| BDI | + | + | 1 | $\mathbb{Z}_2$ |
| D | 0 | + | 0 | $\mathbb{Z}_2$ |
| DIII | − | + | 1 | 0 |
| AII | − | 0 | 0 | $2\mathbb{Z}$ |
| CII | − | − | 1 | 0 |
| C | 0 | − | 0 | 0 |
| CI | + | − | 1 | 0 |

Breaks time reversal



$d = 0$

Add non-entangling terms:

$\Theta h \Theta^{-1} = -h$

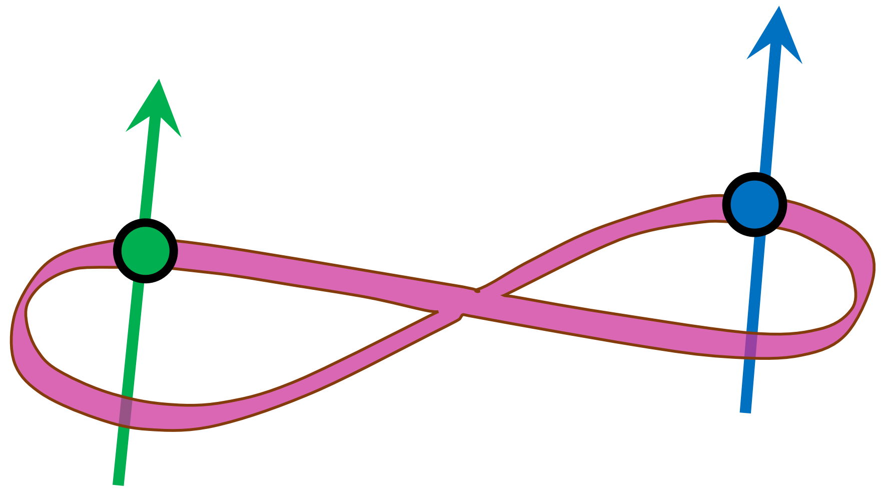
No $\Theta$ $\Theta^2 = "0"$

No $\Pi$ $\Pi^2 = "0"$

# Two qubits in the tenfold classification

| Class | $\Theta$ | $\Pi$ | $S$ | $d=0$ |
|---|---|---|---|---|
| A | 0 | 0 | 0 | $\mathbb{Z}$ |
| AIII | 0 | 0 | 1 | 0 |
| AI | + | 0 | 0 | $\mathbb{Z}$ |
| BDI | + | + | 1 | $\mathbb{Z}_2$ |
| D | 0 | + | 0 | $\mathbb{Z}_2$ |
| DIII | − | + | 1 | 0 |
| AII | − | 0 | 0 | $2\mathbb{Z}$ |
| CII | − | − | 1 | 0 |
| C | 0 | − | 0 | 0 |
| CI | + | − | 1 | 0 |

What is the integer?



$d=0$

$\Theta = (\sigma_y \otimes \sigma_y)K$ $\Theta^2 = +$

No $\Pi$ $\Pi^2 = "0"$

# Computing topological integers

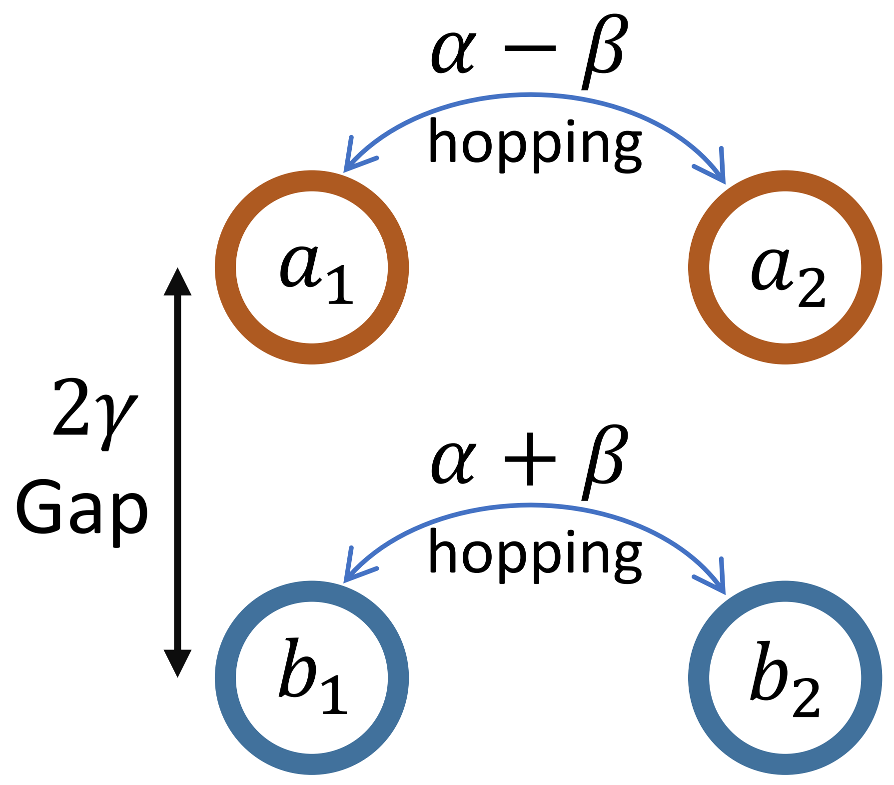
Quantum graphs - Effective tight binding model - topological phase transitions

$$H_R = \alpha\sigma_x \otimes \sigma_x + \beta\sigma_y \otimes \sigma_y + \gamma\sigma_z \otimes \sigma_z$$

Interpret $H_R$ as hopping in a lattice:

$|00\rangle \to$ site $a_1$ $|11\rangle \to$ site $a_2$ $|01\rangle \to$ site $b_1$ $|10\rangle \to$ site $b_2$

$\mathcal{H}_R =$




Orion and Akkermans (2025)

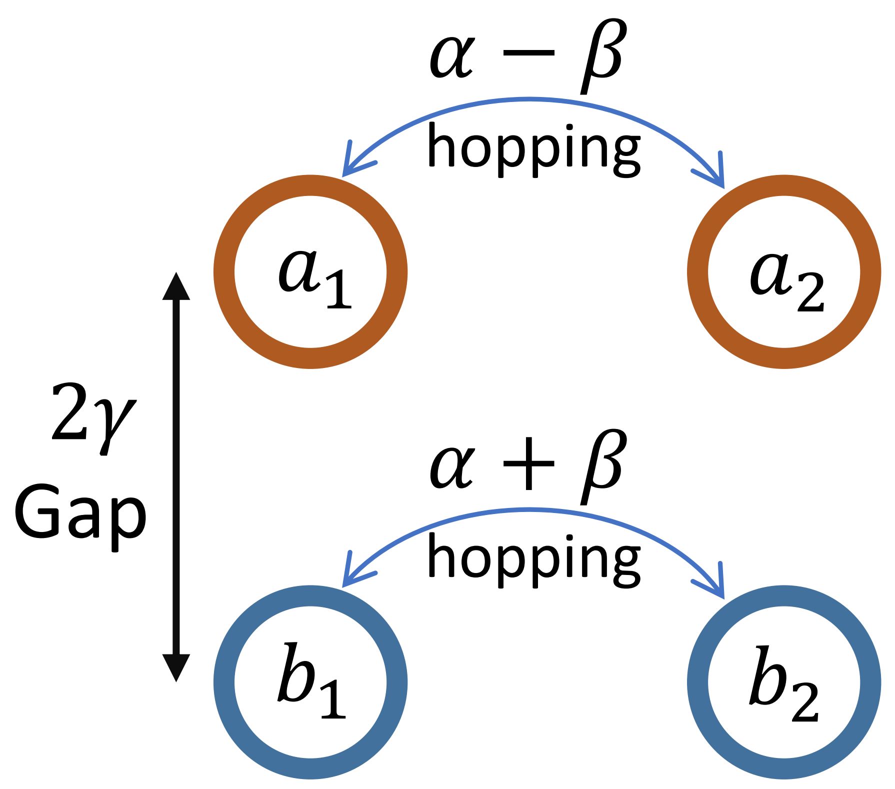
# Computing topological integers

Topological number:
$\nu = \nu_a + \nu_b \in \{0, \pm 1, \pm 2\}$

Finite $\mathbb{Z}$!

$$\nu_a = \begin{cases} 1 & \gamma > \alpha - \beta \\ -1 & \gamma < -(\alpha - \beta) \\ 0 & otherwise \end{cases} \qquad \nu_b = \begin{cases} 1 & \gamma > \alpha + \beta \\ -1 & \gamma < -(\alpha + \beta) \\ 0 & otherwise \end{cases}$$

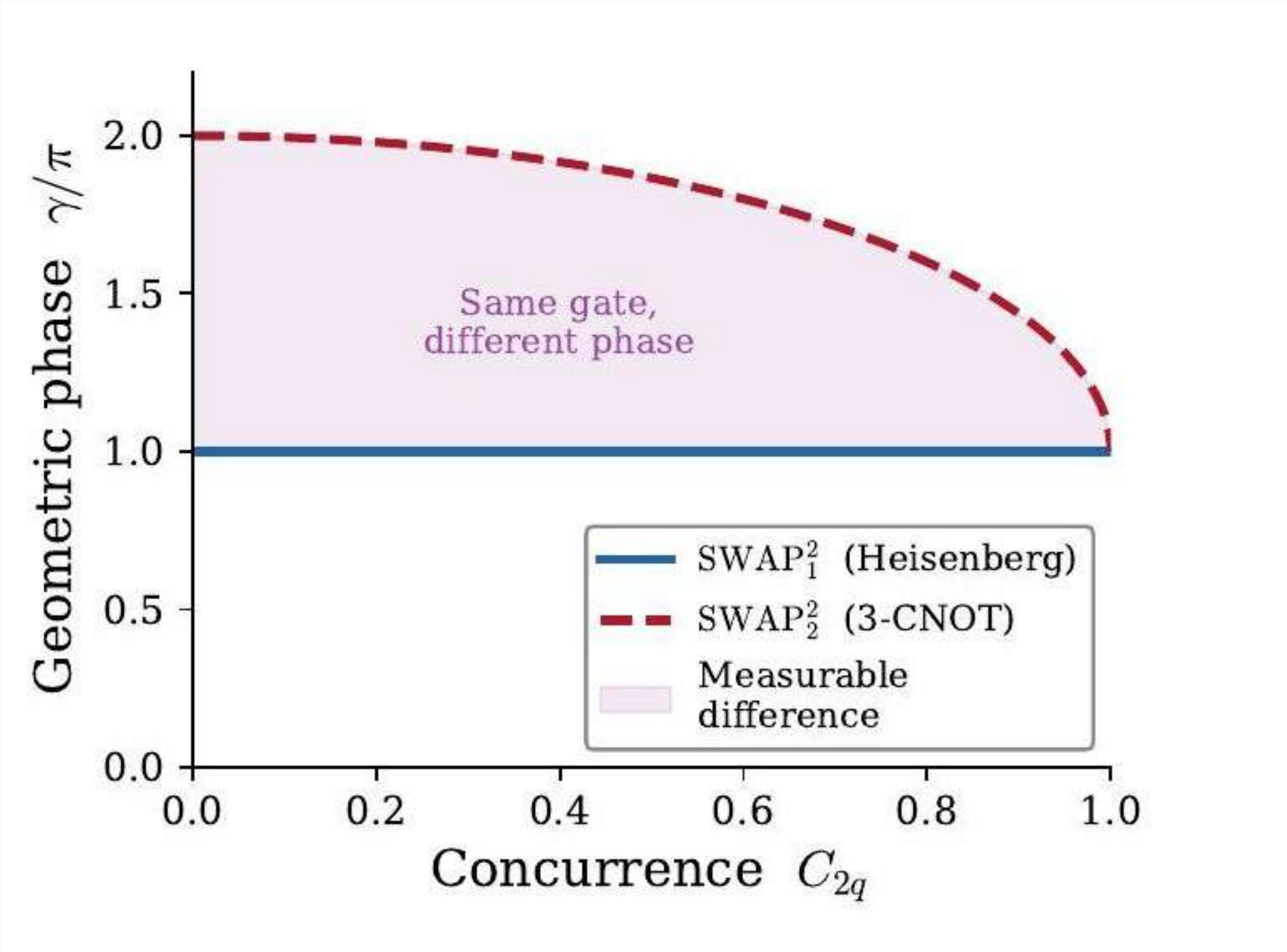
$\mathcal{H}_R =$



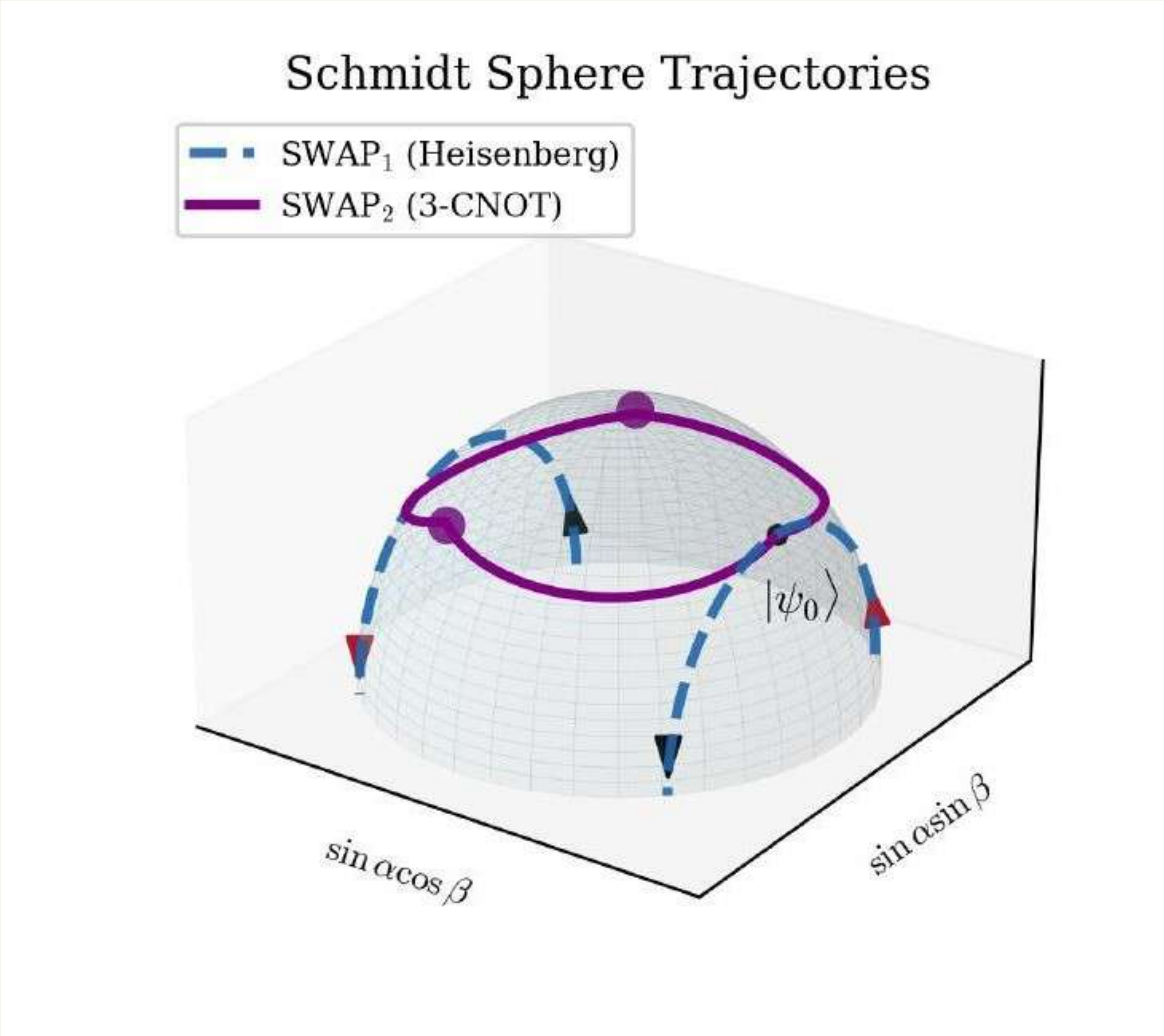

topological phase transitions

Orion and Akkermans (2025)

# Same gate, different topology — measurable




*Geometric phase γ/π vs. concurrence $C_2q$ for two implementations of $SWAP^2$, applied to symmetric initial states.*




*The two implementations trace topologically distinct loops on the Schmidt sphere — different solid angles, different phases.*

***$SWAP_1$ (Heisenberg, vH = +1) vs. $SWAP_2$ (3-CNOT, vH undergoes change): identical unitary, v-distinguishable.***

# Conclusions and Open Directions

Topology is a necessary condition for entanglement.

The Weyl symbol H(k,r) is the right object for topological classification — not the Hamiltonian.

Bell states are zero-dimensional topological zero modes — maximally entangled states at phase boundaries.

## Open directions

- Extension to $N > 2$ qubits via Uhlmann concurrences and multi-qubit Cartan decompositions
- Topological modes in condensed matter (Hubbard model, quantum dot arrays) as protected distinguishable qubits
- Pre-calibration diagnostics: geometric phases identify which topological class is most noise-robust

# Thanks to my research group

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Nadav
Orion

Anna
Hasine

Sean
Faktor

Amit
Goft

Thank you for your attention.

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